THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: David E. Holmgren

TITLE OF THESIS: Analysis of the Light Curves of Eclipsing Variable Stars

DEGREE FOR WHICH THESIS WAS PRESENTED: M. Sc.

YEAR THIS DEGREE GRANTED: 1984

THE UNIVERSITY OF ALBERTA

ANALYSIS OF THE LIGHT CURVES
OF
ECLIPSING VARIABLE STARS

by

© DAVID E. HOLMGREN

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

IN

ASTROPHYSICS

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

SPRING, 1984

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ANALYSIS OF THE LIGHT CURVES OF ECLIPSING VARIABLES submitted by David E. Holmgren in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

Four methods for determining the geometric elements of an eclipsing binary from its light curve are explored in detail. The methods discussed are those of Russell (specifically, the version due to Tabachnik), Kitamura, Kopal (frequency domain approach), and Wood. In each case, the underlying model of an eclipsing binary system is discussed. The various methods of light analysis are then applied to the eclipsing binaries HS Herculis, W Delphini, and HD 219634. The results of each analysis are discussed, and the various methods of analysis are compared with one another. Finally, the relative merits of each model of an eclipsing binary system are considered. Computer programs for the various methods of light curve analysis, along with explanations of their use, are presented in the appendices.

ACKNOWLEDGMENTS

In writing this thesis, the author would like to acknowledge the assistance of several people: Dr. Douglas Hube for his patience and advice, Dr. Austin Gulliver for his assistance, Dr. J.A. Kernahan for his understanding and encouragement, my parents for their encouragement, and finally Mrs. Mary Yiu for her care and patience in the typing of this thesis.

TABLE OF CONTENTS

CHAPTER		PAGE
1	INTRODUCTION	1
2	THE RUSSELL MODEL	4
	2.1 General Principles	4
	2.2 Total and Annular Eclipses	12
	2.3 Partial Eclipses	15
	2.4 A General Formulation	20
	2.5 Non-Sphericity and Rectification	24
	2.6 Differential Corrections	33
	2.7 Conclusion	35
3	THE METHOD OF KITAMURA	36
	3.1 Introduction	36
	3.2 Incomplete Fourier Transforms	37
	3.3 Practical Approach	41
	3.4 Conclusion	43
4	ANALYSIS OF LIGHT CURVES IN THE FREQUENCY DOMAIN	46
	4.1 Introduction	46
	4.2 The Equations of Kopal's Method	47
	4.3 The use of Fourier Series	56
	4.4 The Use of Numerical Integration	61
	4.5 Computing the A_{2m}'s with the Kalman Filter	63
	4.6 Error Analysis for the A_{2m}'s	66

CHAPTER		PAGE
4	(cont'd)	
	4.7 Computing the Elements	68
	4.8 Total and Annular Eclipses of Limb-Darkened Stars	73
	4.9 Partial Eclipses of Limb-Darkened Stars	78
	4.10 Incorporating the Effects of Non-Sphericity	79
	4.11 Conclusions	79
5	WOOD'S MODEL AND THE WINK PROGRAM	81
	5.1 Introduction	81
	5.2 Model Parameters	83
	5.3 Computation of the Elements	89
	5.4 Details of the WINK Program	90
	5.5 The Integration Procedure	90
	5.6 Eclipse Detection and Limit Finding	92
	5.7 The Reflection Effect	97
	5.8 Model Stellar Atmospheres	98
	5.9 Conclusion	99
6	SOME PRACTICAL GUIDELINES	101
7	H S HERCULIS	103
	7.1 Introduction	103
	7.2 Light Curve Analysis	106
	7.3 Conclusions	112

CHAPTER		PAGE
8	W DELPHINI	117
	8.1 Introduction	117
	8.2 Light Curve Analysis	119
	8.3 Conclusions	123
9	HD 219634	125
	9.1 Introduction	125
	9.2 Light Curve Analysis	127
	9.3 Conclusions	129
10	CONCLUSIONS AND COMMENTS	135
	REFERENCES	139
Appendix 1	Russell Model Programs and Rectification	143
	A1.1 Computer Programs	
	A1.2 An Example of Rectification	146
Appendix 2	Program for Kitamura's Method	161
Appendix 3	Programs for Kopal's Method	165
Appendix 4	The WINK8 Program	193
Appendix 5	HD219634-observations	200

LIST OF TABLES

Table	Description	Page
7.1	The geometric elements of H S Herculis.	107
7.2	WINK8 results for H S Herculis.	109
8.1	W Delphini - Irwin's elements.	120
8.2	The geometric elements of W Delphini.	120
8.3	WINK8 - W Delphini.	121
9.1	HD 219634 orbital elements.	133
9.2	HD 219634 orbital elements from WINK8.	134
A1.1		147
A1.2	W Delphini - Observations.	154
A3.1	Input formats for programs using Kopal method.	169
A4.1	Important WINK8 input parameters	196

LIST OF FIGURES

Figure	Description	Page
1	Definition of γ, the angle of fore-shortening.	5
2	Definition of the parameters δ, p, and α.	8
3	Definition of the angle i.	9
4	The geometric relation.	11
5	Eclipse types	16
6	The graphical solution for partial eclipses.	19
7	The phases of first and second contact (θ',θ'').	22
8	The Roche surface.	26
9	A Prolate spheroid.	27
10	The rectification process.	29
11	The angles ε, θ_1, ϕ.	39
12	The definition of A_{2m}.	48
13	Contributions to A_{2m}.	55
14	Orbital parameters in Wood's model.	86
15	Integration grids used in WINK.	93
16	Identification of eclipse types.	95
17	Bisection search for detecting eclipse limits.	96
18	Light curve - H S Herculis.	104
19	Light curve - W Delphini.	118
20	Light curve - HD 219634.	126

CHAPTER 1

INTRODUCTION

Eclipsing binary stars are, in many ways, very informative to the astronomer and astrophysicist. The study of eclipsing binary stars can reveal a great deal about the sizes of stars, their masses, densities, and internal structure. Such information is also valuable in ascertaining the evolutionary state of the two (or more) stars constituting an eclipsing binary system. In certain cases in which the two component stars of an eclipsing binary system are in close proximity to one another, it is also possible to deduce the amount of tidal distortion present, and to check for the presence of matter streams between the two stars. With all of this information in hand, a scale model of an eclipsing binary star may be constructed, and hence, our knowledge of the system will be complete. One may then use this scale model to look for long-term effects such as apsidal motion, the presence of which can be deduced from a secular change in the period of the eclipsing binary. Such measurements may also be used to verify Einstein's theory of General Relativity.

The key problem, however, is in interpreting the observed light changes (the 'light curve') of an eclipsing binary. This is the critical step which lies between making the observations and coming to the conclusions outlined in the previous paragraph. To this end a great deal of work has been done, from the first tentative steps taken by Russell in 1912 (Russell, 1912), which dealt with the determination of the geometric elements (the relative radii of the stars, the orbital inclination angle, and the relative luminosities) of an eclipsing binary consisting of non-limb darkened spherical stars, to the recent Fourier analysis methods of Kopal (see for instance, Kopal, 1979), which use the harmonic content of the observed light changes to deduce the geometric elements.

A necessary ingredient in all methods for determining the geometric elements from the observed light changes is a realistic physical model of the binary star, preferably involving as few assumptions as possible regarding the forms of the stars (i.e., spherical, non-spherical) and their physical properties (temperature, luminosity, etc.). The complexity, and consequently the realism, of such models of eclipsing binaries has grown since the initial investigation of the problem by Russell in 1912. Present day models of eclipsing binaries describe a range of situations, from that of two well-separated spherical stars to systems in which both stars are in contact, in which case both stars are greatly distorted by their mutual tidal interaction. A question of some importance is then: which of the several models of eclipsing binary stars currently available best describes a given eclipsing binary star? To answer this question, representative models of eclipsing binary stars and their accompanying methods of light curve analysis will have to be analyzed and the appropriate conclusions drawn.

There are three broad categories of methods used in light curve analysis. They are the "classical" or "Russell-type" methods previously referred to, the "synthesis" methods, devised in the early 1970s, and the "frequency-domain" methods of Kopal previously referred to. Two methods of the "classical" type are the Russell-Merrill (1952) method and the method of Kitamura. A fine example of a "synthesis" method is a method devised by Wood. Several versions of Kopal's frequency-domain method exist, but the best of these are the most recent ones (e.g., Kopal (1982)). These methods of light curve analysis will be applied to three stars covering a wide range of physical conditions, from the well-separated case of W Delphini, to the case of HS Herculis with its matter stream, and finally to HD 219634, which may be a massive binary and possibly even an X-ray source (see Gulliver, Hube and Lowe (1982)). This analysis will, we hope, shed some light on the validity of the various models of eclipsing binary stars to be considered.

CHAPTER 2

THE RUSSELL MODEL

2.1 General Principles

The first steps toward an interpretation of eclipsing binary light curves were taken by Russell in 1912 (Russell (1912a,b)). Subsequent refinements to the theory were made by Russell and two collaborators, Merrill and Shapley (see Russell and Shapley (1912); Russell and Merrill (1952)). The Russell model can be applied to both spherical and non-spherical stars with varying degrees of accuracy.

The "spherical model" assumes that the eclipsing binary system consists of two spherical stars moving in circular orbits about a common centre of gravity. The distribution of surface brightness $J(\gamma)$ over the disk of each star is assumed to follow the "cosine law"

$$J(\gamma) = J(0)(1 - x + x\cos\gamma) \qquad (2.1)$$

where $J(0)$ is the surface brightness at the centre of the observed disk of either star, x the coefficient of limb darkening, and γ the angle of foreshortening, or the angle between the line of sight and a radius vector from the centre of the star (see figure 1). The angle γ varies between 0 and 90 degrees. At this point, it should be noted that the physical properties of the stars enter the Russell model only through equation (2.1). The detailed features of the stars (e.g., starspots, magnetic fields,

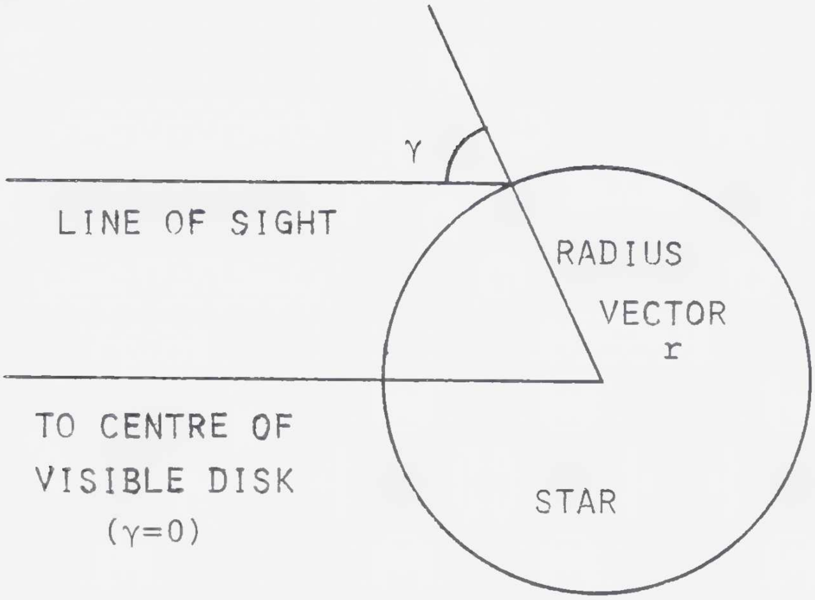

FIGURE 1. DEFINITION OF γ, THE ANGLE OF FORESHORTENING.

intrinsic variability) are not accounted for.

To arrive at a method for determining the geometric elements of an eclipsing binary using the spherical model, it will be necessary to consider, in some detail, the geometry of an eclipse. The treatment that follows is similar to that given by Irwin (1962) or Russell (1912a,b). Before plunging headlong into this problem, it will be necessary to define some of the quantities that will be used. The radius of the smaller star is denoted by r_s , and that of the larger star by r_g. These radii are measured in units of the centre-to-centre separation of the component stars of an eclipsing binary. The relative luminosity of the small and large stars will be denoted by L_s and L_g , respectively. These luminosities are so defined that

$$L_s + L_g = 1 \ . \qquad (2.2)$$

Since an eclipsing binary light curve displays brightness as a function of time, it is necessary to define an orbital phase θ by

$$\theta = \frac{2\pi}{P}(t - t_o)$$

where P is the period of the eclipsing binary, t the time at which the brightness was observed or is to be calculated, and t_o the time of conjunction, which usually coincides with the time of minimum light during the primary (deeper) eclipse. The times t and t_o are measured in Julian days, while the period P is measured in days. With explicit reference to the eclipse, it is customary to define three additional quantities, namely k, p, and δ. The dimensionless parameter k is simply the ratio of the radii r_s and r_g:

$$k = \frac{r_s}{r_g} \quad , \quad (0 \le k \le 1) \ .$$

The "geometric depth" p is another dimensionless parameter, which represents the extent to which the eclipse has progressed at any eclipse phase θ. A parameter closely related to p is δ, the apparent separation of the centres of the disks of the stars. The parameters p and δ are shown in figure 2. The following equation gives the relationship between p and δ (and does, in fact, serve to define p)

$$p = \frac{\delta - r_g}{r_s} \qquad (2.3a)$$

or

$$\delta = r_g(1 + kp) \quad . \qquad (2.3b)$$

using the definition of k stated above. Relation (2.3b) is the more useful of the two relations relating p and δ. The quantity δ may also be related to the phase θ and the orbital inclination. The orbital inclination is defined as the angle between a plane perpendicular to the line of sight (the "celestial sphere") and the orbital plane (see figure 3). Through the use of some simple trigonometry, and recalling that the two stars constituting the eclipsing binary have a unit separation, one arrives at the "geometrical relation":

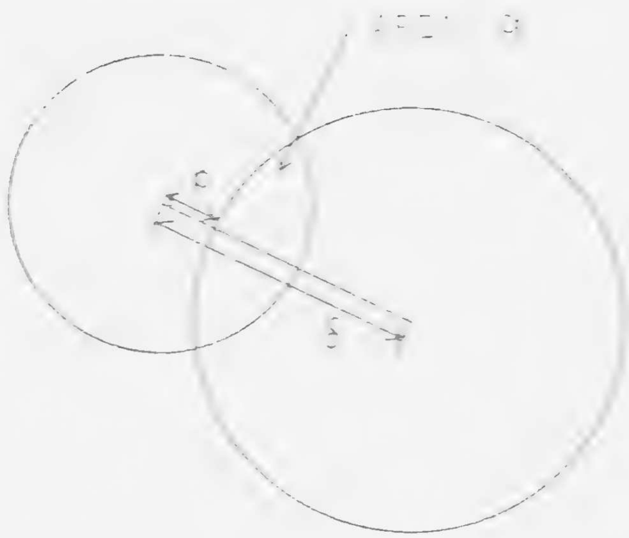

FIGURE 2. [illegible]

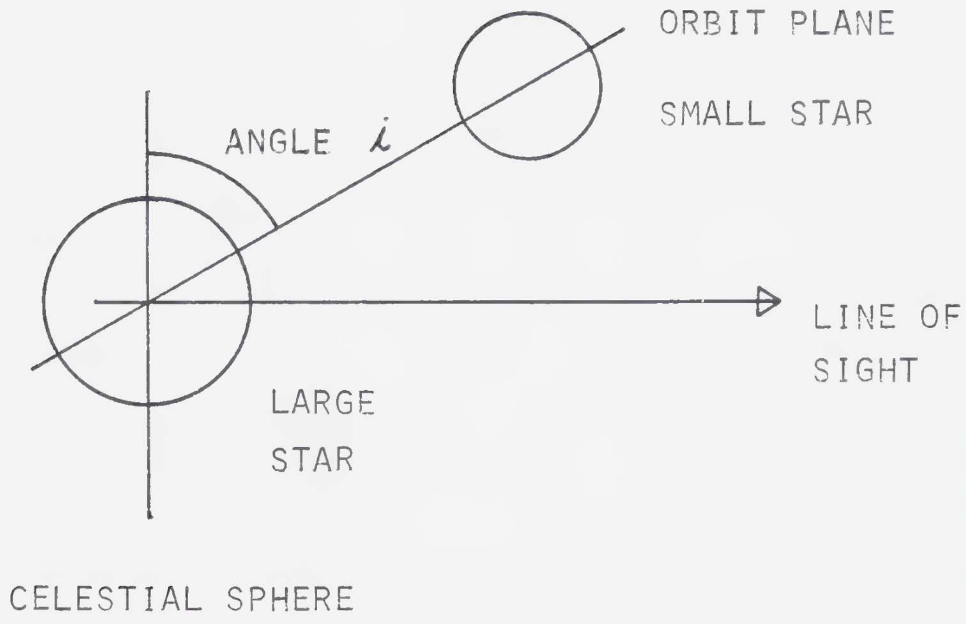

FIGURE 3. DEFINITION OF THE ANGLE i.

$$\delta^2 = \cos^2 i + \sin^2\theta \sin^2 i = 1 - \sin^2 i \cos^2\theta$$

or

$$r_g^2(1+kp)^2 = \cos^2 i + \sin^2\theta \sin^2 i \quad . \tag{2.4}$$

Equation (2.4) is the fundamental equation of all Russell-type methods for the determination of the geometric elements. The derivation of the geometric relation is outlined in figure 4. It should also be noted that only a relative orbit is considered, namely, the relative orbit of the smaller star about the larger one. In the case of an elliptical orbit, the geometric relation would be multiplied by R^2, R being the separation between the stars at any orbital phase. The orbital phase would have to be replaced by $v-\omega$, where v is the true anomaly and ω the angle between the line of apsides and the line of sight. The vast majority of eclipsing binary systems have circular orbits, however, largely as a consequence of their short orbital periods and consequent tidal interations.

As an aid in the interpretation of eclipsing binary light curves, a relative luminosity ℓ is defined. This luminosity is related to a change in magnitude Δm by

$$\ell = 10^{-0.4\,\Delta m} \tag{2.5}$$

where Δm is taken relative to the magnitude of the eclipsing binary just before the start of the eclipse. The value of ℓ at the minimum of the eclipse is denoted by λ. A quantity $\alpha = \alpha(k,p)$ is also defined; representing the fractional

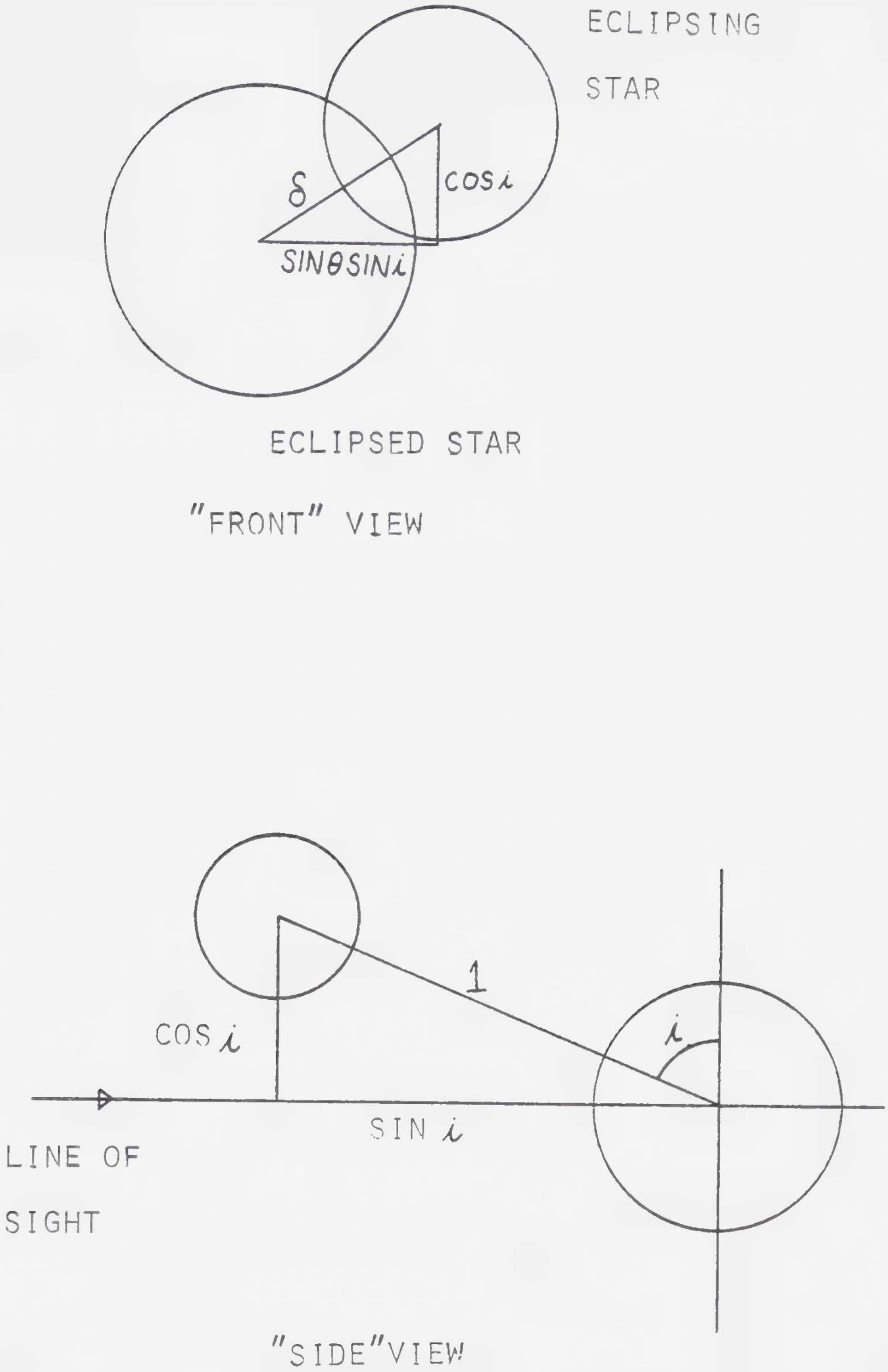

FIGURE 4. THE GEOMETRIC RELATION.

light loss. At any eclipse phase (see figure 2). This quantity may be determined directly from the observations by

$$\alpha = \frac{1 - \ell}{1 - \lambda} \tag{2.6}$$

$$\ell = 1 - \alpha L \quad .$$

2.2 Total and Annular Eclipses

We are now in a position to describe Russell's method for the determination of r_s, r_g, and i. The following derivation may also be found in a recent book by Kopal (1979, pg. 110). The basic idea of the method is to write down the geometric relation (eqn. (2.4)) for three eclipse phases and to consider $\sin^2 i$, $\cos^2 i$, and r_g^2 as the unknowns. For the system of equations to have a unique solution,

$$\begin{vmatrix} \sin^2\theta_1 & (1+kp_1)^2 & 1 \\ \sin^2\theta_2 & (1+kp_2)^2 & 1 \\ \sin^2\theta_3 & (1+kp_3)^2 & 1 \end{vmatrix} = 0 \quad . \tag{2.7}$$

In this equation, k is the only unknown, since p can in principle be determined from $\alpha(k,p)$. The phases θ_1 and θ_2 are chosen so as to correspond to $\alpha = 0.6$ and $\alpha = 0.9$ respectively. From this point on, the method used to determine r_s, r_g and i depends on the type of eclipse. It should also be noted that the entire light curve is not required for the analysis. Only one half of an eclipse is required.

For a total or annular eclipse (see figure 5), the determinant in equation (2.7), is written as

$$\sin^2\theta_3 = A + B\,\psi(k,p) \qquad (2.8)$$

where

$$A = \sin^2\theta_1 \quad , \quad B = A - \sin^2\theta_2$$

and

$$\psi(k,p,\alpha) = \frac{2(p_3 - p_1) + k(p_3^2 - p_1^2)}{2(p_1 - p_2) + k(p_1^2 - p_2^2)} \quad . \qquad (2.9)$$

If $\sin^2\theta_3$ is allowed to represent any eclipse phase, then A and B may be determined, and finally $\psi(k,p,\alpha)$ for the given θ_3. Thus, one tabulates $\psi(k,p,\alpha)$ for all eclipse phases. $\psi(k,p,\alpha)$ can also be tabulated using equation (2.9), so a comparison between the observed and theoretical values of $\psi(k,p,\alpha)$ can be made, allowing k to be determined for each eclipse phase. Russell tabulated $\psi(k,\alpha)$ for both types of eclipse, but the most comprehensive tabulation was that of Russell and Merrill (1952). A shorter and more useful set of tables (for $x = 0.5$) was published by Irwin (1962). To summarize, one finds a value for k by determining $\psi(k,\alpha)$ from the light curve by equation (2.8), and by using these observed values of $\psi(k,\alpha)$, along with the corresponding values of α determined by equation (2.6), to do inverse interpolation in a table of $\psi(k,\alpha)$, thereby producing a range of values for k. An average of the values of k is taken, and this number, $\langle k \rangle$, is then taken to be the 'correct k' in later calculations. The inclination i and the radius r_g can now be found from

$$\cot^2 i = \frac{B}{\phi_2(k)} - A \quad \text{and} \quad (r_g \csc i)^2 = \frac{B}{\phi_1(k)} , \qquad (2.10)$$

where $\phi_1(k)$ and $\phi_2(k)$ are two auxiliary functions also tabulated by Russell and Merrill in the reference quoted above. The value of r_s can now be found by using the definition of k.

The method of finding r_s, r_g, and i just described was modified by Russell and Merrill (1952) to use more points on the light curve during an eclipse. This is achieved by using three weighted means of $\sin^2\theta$ and $\psi(k,\alpha)$, and by defining a new function R(x,k)

$$R(x,k) = \frac{M_1[\sin^2\theta] - M_2[\sin^2\theta]}{M_2[\sin^2\theta] - M_3[\sin^2\theta]} = \frac{M_1[\psi] - M_2[\psi]}{M_2[\psi] - M_3[\psi]} , \qquad (2.11)$$

$$M_j[\sin^2\theta] = A + B\,M_j[\psi] , \quad j = 1,2,3$$

where $M_j[\sin^2\theta]$, $j = 1,2,3$, is a weighted mean of $\sin^2\theta$ for certain values of α, and $M_j[\psi]$, $j = 1,2,3$, is the corresponding weighted mean of $\psi(k,\alpha)$. Only one table is required to find k given R(x,k), $M_1[\sin^2\theta]$, $M_2[\sin^2\theta]$, and $M_3[\sin^2\theta]$. The values of r_s, r_g and i are obtained as in the earlier version of the Russell method. The version just described, known as the 'Russell-Merrill' method, has the advantage of simplicity and greater computational speed, and will be used in subsequent examples.

2.3 Partial Eclipses

In the case of a partial eclipse (see figure 5), a different approach is required. The problem is more difficult to solve since observations of both eclipses are required for a unique solution, and the value of α at mid-eclipse (denoted by α_o) is also unknown. At mid-eclipse, the geometric relation (2.4) becomes

$$\cos^2 i = r_g^2 (1 + kp_o)^2 \quad , \quad p_o = p(k,\alpha_o) \qquad (2.12)$$

since $\theta = 0$ at this point. Upon subtracting this result from the geometric relation, one has

$$\sin^2\theta = (r_s r_g \csc^2 i)[2(p - p_o) + k(p^2 - p_o^2)] \ . \qquad (2.13)$$

This is the fundamental equation for the analysis of partial eclipses. Russell's approach (Russell, 1912b) was to first define n as the ratio of $1-\ell$ to $1-\lambda$ at any eclipse phase, and to take the value of $\sin^2\theta$ at $n = 0.5$ as a 'base point'. Russell then divided equation (2.13) by its counterpart at $n = 0.5$, obtaining

$$\frac{\sin^2\theta(n)}{\sin^2\theta(0.5)} = \frac{2(p - p_o) + k(p^2 - p_o^2)}{2(p_1 - p_o) + k(p_1^2 - p_o^2)} \equiv \chi(k,\alpha_o;n) \qquad (2.14)$$

where $\theta(0.5)$ denotes the value of θ when $n = 0.5$, $\theta(n)$ the value of θ for any other n, and p_1 the value of p at $n = 0.5$. In the analysis of partial eclipses, the χ-functions play a similar role as do the ψ-functions in the analysis of total and annular eclipses. However, in the partial eclipse case,

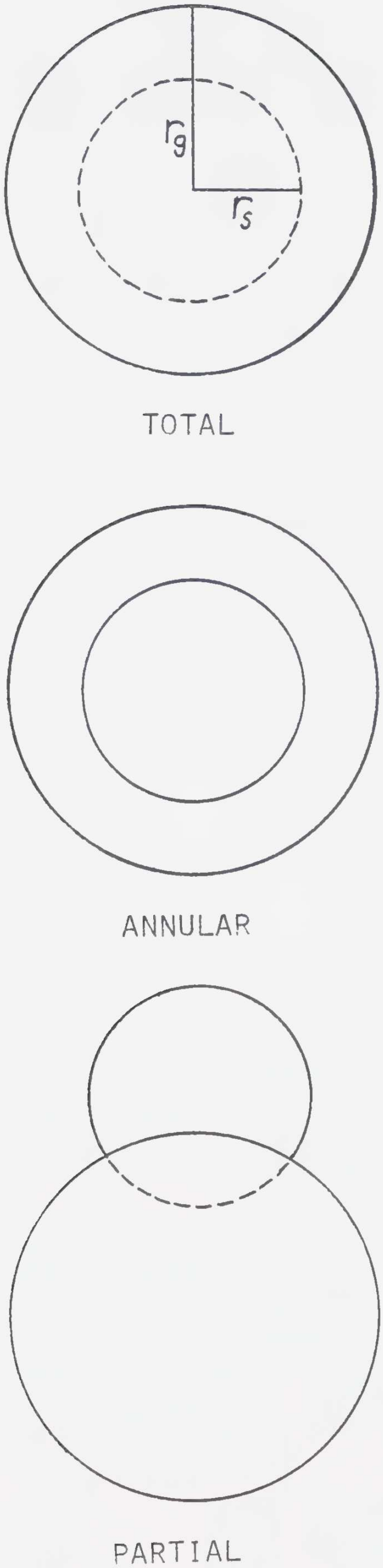

FIGURE 5. ECLIPSE TYPES.

the solution is graphical. The values of $\chi(k,\alpha_o;n)$ may be computed from the light curve by using the left-hand side of equation (2.14). Since k and α_o are to be solved for first, only two values of χ are needed. Suppose these values to be denoted by $\chi = c_1$ and $\chi = c_2$. One may also compute χ from the right-hand side of equation (2.14), and therefore tabulate $\chi(k,\alpha_o;n)$. The most complete tables of $\chi(k,\alpha_o;n)$ are those compiled by Russell and Merrill (1952), which can be used for both occultation and transit eclipses and any value of limb darkening x. After choosing a value of x, one uses the χ-tables to plot α_o as a function of k for the given values of χ, namely c_1 and c_2. Therefore, the point at which these curves intersect should provide the required values of k and α_o. Unfortunately, the solution obtained is indeterminate since it is not known whether the given eclipse is an occultation or a transit. Therefore, both eclipses must be used. The type of eclipse may now be determined quite easily by using the relationship (see Irwin (1962), p. 607)

$$\chi^{oc}(k,\alpha_o;n=0.8) > \chi^{tr}(k,\alpha_o;n=0.8) \qquad (2.15)$$

where 'oc' denotes occultation and 'tr' transit. This relation may be verified by consulting the appropriate tables of χ for $n = 0.8$. Another problem arises in the fact that the intersection of the two χ = constant curves can be quite shallow, resulting in an indeterminate solution once again. To remove this indeterminacy, another independent relation must be introduced.

If λ denotes the value of minimum light for either eclipse, then

$$\lambda = 1 - \alpha_o L \qquad (L_s + L_g = 1)$$

where L is the relative luminosity of either star. If one writesthis out for both eclipses and solves for α_o, there results

$$\alpha_o^{oc} = 1 - \lambda_a + \frac{1 - \lambda_b}{k^2 Y} \qquad \text{for an occultation}$$

(2.16)

$$\alpha_o^{tr} = 1 - \lambda_b + (1 - \lambda_a) k^2 Y \qquad \text{for a transit}$$

where λ_a and λ_b represent λ for occultation and transit eclipses respectively and Y denotes the ratio $\alpha_o^{oc}/\alpha_o^{tr}$. Either of equations (2.16) is known as a "depth" equation, since such equations relate the depth of an eclipse $(1-\lambda)$ to α_o and k. Equations (2.16) are incorporated in the solution method for partial eclipses by obtaining values of k and α_o for successive values of Y. Tables of $Y(\alpha_o, k)$ exist (Irwin (1962) gives tables of $q_o(k, \alpha_o^{oc}) = k^2 Y(k, \alpha_o^{oc})$) for this purpose. The set of values of k and α_o so obtained are plotted on the same graph as are the equations for χ = constant mentioned earlier. The curve so defined will usually make a steep intersection with the χ = constant curves, thereby rendering the solution determinate. An example of such a graph is shown in figure 6.

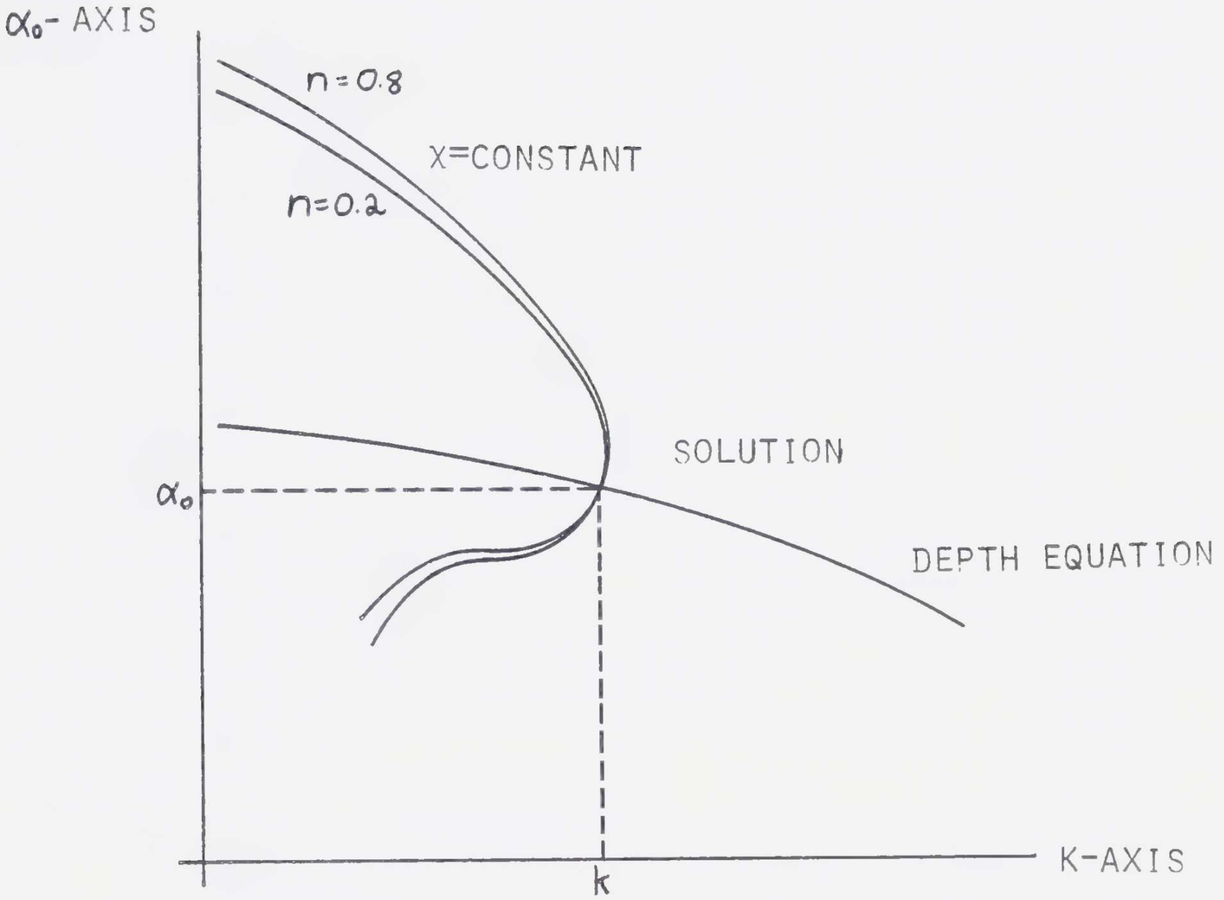

FIGURE 6. THE GRAPHICAL SOLUTION FOR PARTIAL ECLIPSES (ADAPTED FROM IRWIN(1962, PG. 607)).

The preceding paragraphs have described the form of the Russell-Merrill method that is most useful for the analysis of total and annular eclipses, but which for partial eclipses is not the best nor the most useful approach. In fact, Kopal (Kopal, 1979, pp. 113-114) has argued strongly against the use of the χ-functions for determining the orbital elements. The essence of his criticism is that the position of any 'fixed' or 'base' points, as used in the Russell-Merrill method, can be determined only to some finite accuracy, and that this uncertainty would propagate through the entire light curve solution, leading to an uncertainty in the values of r_1, r_2 and i. Furthermore, the solution is fitted only at the 'fixed points', not at all of the data points. One could imagine a "worst case" in which a rather large initial error would propagate and magnify through the solution, leading to wildly erroneousresults. It is situations such as these which have led other workers to use other versions of the Russell-Merrill method.

2.4 A General Formulation

The Russell-Merrill method may be restated in a form useful for any type of eclipse, and moreover, in a form that is amenable to use with electronic computers. This method, due to Kopal (see Kopal (1979), pg. 115), takes the geometric relation and rewrites it in the form $y = ax + b$, which is linear. If one defines

$$x = \sin^2\theta \quad \text{and} \quad y = (1 + kp)^2$$

then the geometric relation may be written in the form

$$x = (r_g \csc i)^2 y - \cot^2 i$$

or

$$y = \frac{\sin^2 i}{r_g^2} x + \frac{\cos^2 i}{r_g^2} \quad , \qquad (2.17)$$

this latter form being suggested by Tabachnik (1973). The elements r_g, r_s, and i may be found from the following equations, where $a = \sin^2 i / r_g^2$ and $b = \cos^2 i / r_g^2$:

$$\tan^2 i = \frac{a}{b} \quad , \quad r_g = (a+b)^{-1/2} \quad \text{and} \quad r_s = kr_g \ . \qquad (2.18)$$

If an initial value of k is used to determine p from a table of $\alpha(k,p)$, then the correct value of k will be the one that renders equation (2.17) a straight line. The straight line is fitted using the standard least-squares techniques. A good initial guess at k can be made in several ways. The simplest is to use the formula

$$k = \frac{\theta' - \theta''}{\theta' + \theta''}$$

where θ' is the phase angle of first contact and θ'' the phase angle of second contact (see figure 7 for definitions of θ' and θ''). Other methods are given in Appendix 1. The advantage in using equation (2.17) lies in the fact that all available eclipse data are used, and no special points on the light curve are required. Moreover, one can use any $\alpha(k,p)$ table, for either an occultation or a transit, and for any limb darkening to determine $p(k,\alpha)$. Therefore, this version of the Russell method is clearly the preferable one.

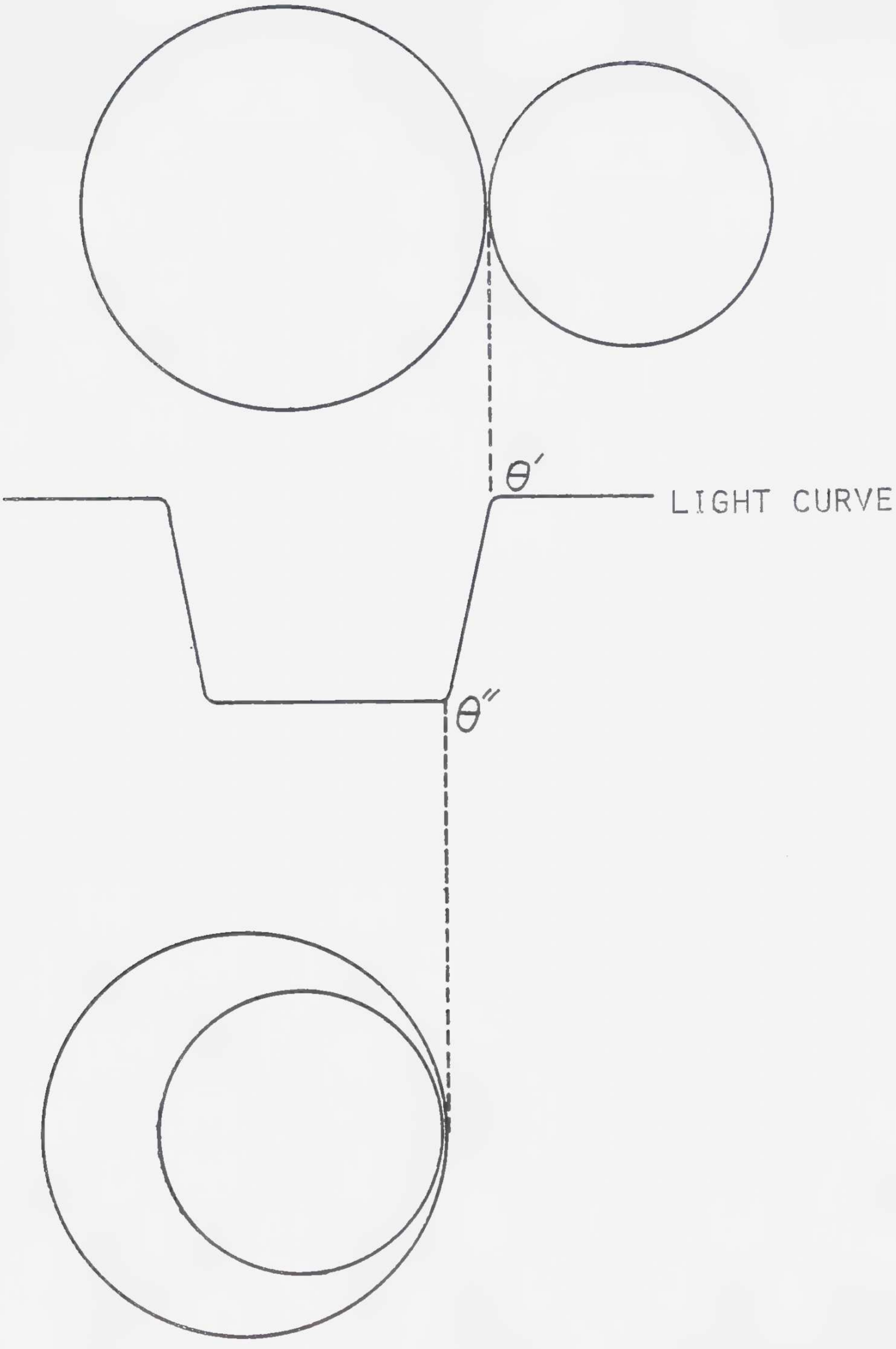

FIGURE 7. THE PHASES OF FIRST AND SECOND CONTACT (θ´,θ´´).

The present author has written a program for the Tl-59 programmable calculator to use equation (2.17) in the analysis of eclipsing binary light curves. The values of $\sin^2\theta$ and p are used as input. The Tl-59 calculator is particularly convenient since it has a built-in least-squares linear fit routine, which can be easily incorporated into a larger program. This program will be used in later sections when particular stars are considered. A listing of the program is presented in Appendix 1. A computer program, LINE, incorporating Tabachnik's method, is also listed in Appendix 1.

Before discussing the application of the Russell method and the Russell-Merrill method to close eclipsing binary stars, it should be mentioned that several other versions of the Russell-Merrill method exist in the literature. Most of these are due to Kopal, in particular the iterative methods (based on equation (2.13)), which have proven to be very useful. These methods are conveniently summarized in the 1979 book by Kopal. A computer program incorporating an iterative method has been published by Jurkevich (1970). An important variation due to Kitamura (1965), which employs Fourier transforms of the light curve, will be considered in the next section. Some methods which are no longer in use are those of Scharbe (1925) and Fetlaar (1923). The latter method is summarized in a book by Tsesevich (1973), which also contains a description of a method called the 'express method'. A recent revival of Kopal's iterative methods may be found in Look et al. (1978), which also contains an interesting application of the depth equation.

2.5 Non-Sphericity and Rectification

Naturally, one cannot apply the Russell-Merrill method (or any one version of it) to all eclipsing binary stars. Not all eclipsing binary stars have spherical component stars since there are inevitably tidal effects in any close system. Those eclipsing binaries with relatively short periods, less than about 3 days, will most certainly have some tidal distortion present, since the two stars involved will be quite close to one another (Kepler's harmonic law: $P^2 \propto a^3$). There are often other associated effects. An obvious one is that one star will heat the other, the effect being a mutual one. When first discovered, this effect was called the "reflection effect", since it was believed at the time that light from one star was reflecting off the surface of the other: Though inaccurate, the name stuck. As the theory of stellar atmospheres evolved beyond the well-known "gray" case, it was realized that the "reflection" effect was really a heating effect. The reflection effect has become one of the most difficult effects to understand, and hence model, since the magnitude of the effect depends not only on the closeness of the stars, but also on the state of their atmospheres. The problem as it currently stands is summarized by Sahade and Wood (1978). A comprehensive study of the reflection effect, typical of many done, is that done by Napier (1968).

Another effect present in eclipsing binaries is a direct consequence of the closeness of the component stars. This is the presence of streams of matter between the stars. Matter streams can arise in two ways, the first being the presence of one star with a moving atmosphere (stellar wind). Wolf-Rayet stars and red giant stars can be involved in this type of mass exchange. A mass exchange can also arise if one star expands out to its Roche limit (see figure 8). Some of the matter from the expanding star is then drawn off by the other star (through its gravitation), with the result being either an accretion disk or a "hot spot", where the matter stream makes contact with the atmosphere of the attracting star. In the Russell model, the effects of tidal interaction, reflection, and mass transfer are dealt with by the process of rectification.

The dynamical and physical theory upon which the process of rectification rests will not be developed here. A comprehensive treatment may be found in an article by Martynov (1973) in the book edited by Tsesevich (1973). The treatment to be followed here is that given by Proctor and Linnell (1972). The Russell model treats the stars of a close eclipsing binary as prolate spheroids (see figure 9), although the results obtained at the end of the rectification process can be converted into results applying to a triaxial ellipsoid. The object of rectification is to convert the light curve of an eclipsing binary consisting of distorted stars into an equivalent "spherical" light

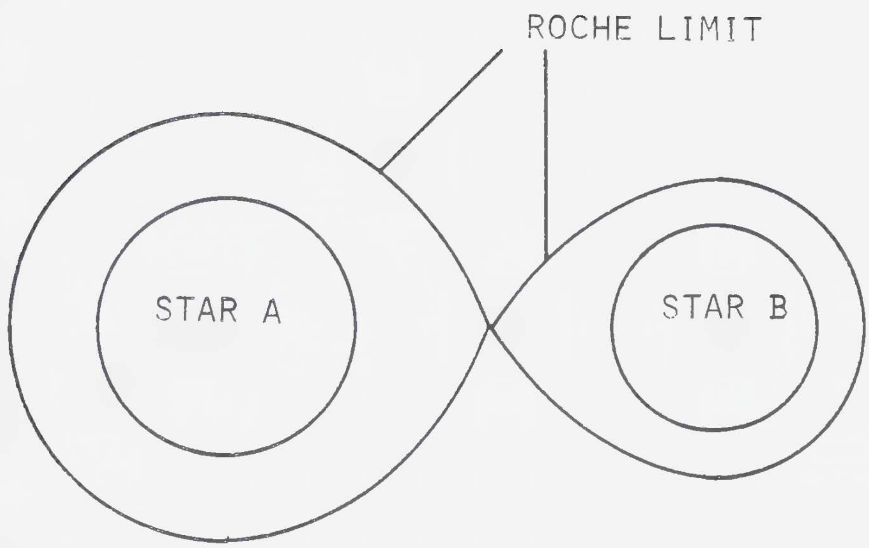

FIGURE 8. THE ROCHE SURFACE.

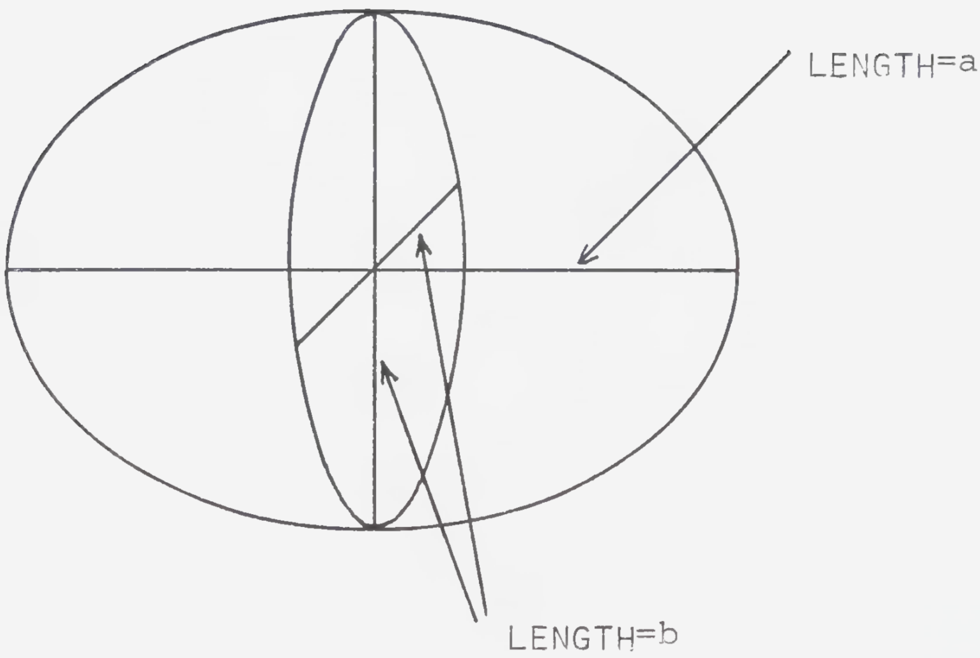

FIGURE 9. A PROLATE SPHEROID.

curve. Thus, rectification produces a rectified luminosity ℓ_r and a rectified phase θ_r, given unrectified values of ℓ and θ. More rigorously, if an observer at point 0, having direction cosines ℓ, m, n is watching an ellipsoidal star in a close binary system, then the process of rectification is an affine transformation that carries the observer at 0 to another point 0', with direction cosines ℓ', m', n', at the same distance from the sphere. Since the transformation carries an ellipsoid into a sphere (with a radius equal to the ellipsoid's semi-major axis), then the luminosity of the spherical star, as seen at 0', must be the same as that seen from the ellipsoid at 0. Since the light from the ellipsoid varies with phase, the light from the sphere must be modulated to produce the same light variation. The affine transformation is illustrated in figure 10. It was shown by Russell and Merrill (1952) that the light variation from a prolate spheroid with axes (a,b,b) is the same as that from a triaxial ellipsoid having axes (a,b,c). If the orbit has an inclination j in the prolate spheroid case and i in the ellipsoid case, then the semiaxes b and c of the triaxial ellipsoid are related by

$$\frac{\tan^2 j}{\tan^2 i} = \frac{c^2}{b^2} \quad . \tag{2.19}$$

Before proceeding further, it is necessary to define some parameters that will be used in the discussion that follows. The oblateness ε is defined as (a-b)/a, a and b being the semiaxes of either the prolate spheroid or the

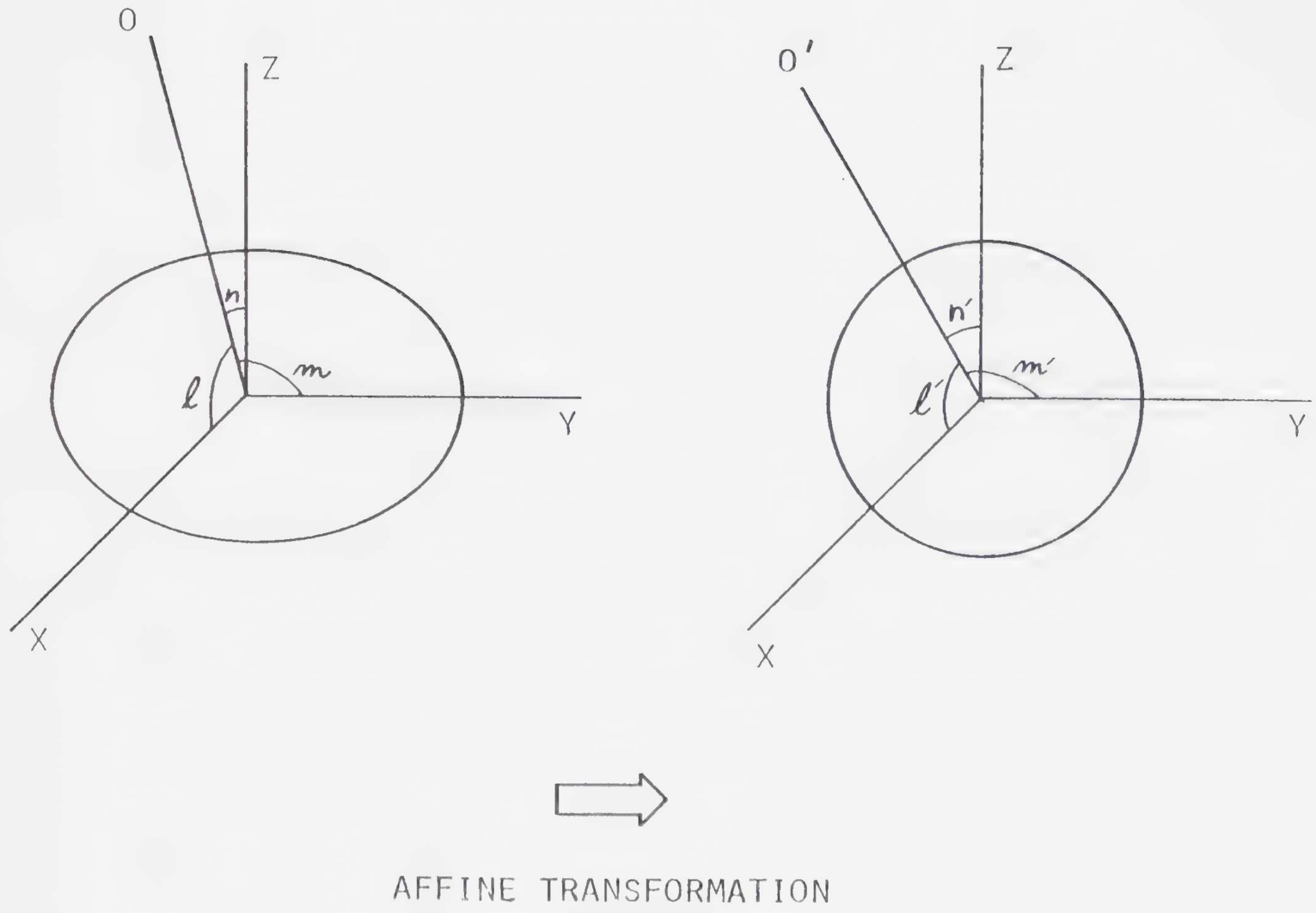

FIGURE 10. THE RECTIFICATION PROCESS.

triaxial ellipsoid. An approximate expression for ε is $\frac{1}{2}\eta^2$, η being the eccentricity of the cross-section of the star in the orbital plane. It will also be useful to define z, which is equal to $2\varepsilon \sin^2 j$. A non-spherical star will not have a uniform surface gravity, and consequently, those parts of the surface of the star farther from the star's center will appear cooler, while those parts closer to the center (near the pole of rotation) will appear hotter. A quantity that describes this effect is the gravity darkening coefficient y, which is defined by Martynov (1973):

$$y = \frac{c_2}{4\lambda T_o(1 - e^{c_2/\lambda T_o})} \qquad (2.20)$$

where λ is the wavelength of observation, T_o the surface temperature, and c_2 (equal to hc/k) is a numerical constant whose value depends on the units of λ and T_o (see Gray (1976), pg. 117). Therefore, the observed intensity at any point on the star's surface will be

$$I = H(1 - x + x \cos\gamma)(1 - y + y g/g_o) \qquad (2.21)$$

where H is the intensity at the centre of the observed disk, x and γ are defined in equation (2.1), g is the surface gravity at any point on the star, and g_o is a reference value of g, usually taken to be the value at the pole of the star. The light from either star can be expressed as (Russell and Merrill (1952), pg. 42):

$$\ell(\theta) = I(90^{\circ})(1 - N\varepsilon \sin^2 j \cos^2\theta) + G f(\phi) \qquad (2.22)$$

where

$$N = \frac{15 + x}{15 - 5x}(1 + y) \ .$$

In this formula, $I(90^{\circ})$ is the light from one star at quadrature phase ($\theta = 90^{\circ}$), G is an 'albedo' factor that determines the fraction of light received from the companion star which is reradiated at the wavelength of observation (Russell and Merrill (1952), pg. 46), and $f(\phi)$ is a 'phase function' that characterizes the reflection effect. The procedure of rectification is one in which the effects of reflection and ellipticity are removed by writing out the equation for $\ell(\theta)$ for both stars, finding the sum of these two equations, and then doing the appropriate subtraction and division to produce the value of $\ell(\theta)$ for a system consisting of spherical stars. In practice, rectification is done by fitting a Fourier series of the form:

$$\ell(\theta) = A_o + A_1 \cos\theta + A_2 \cos 2\theta + B_1 \sin\theta + B_2 \sin 2\theta \qquad (2.23)$$

to the light curve of the eclipsing binary outside the eclipses (one may use $\cos^2\theta$ and $\sin^2\theta$ instead of $\cos 2\theta$ and $\sin 2\theta$, by making use of a trigonometric identity, but the coefficients will then take on different meanings). The series may be fitted either by the least-squares method, or by a graphical method developed by Russell and Merrill. An example of the latter may be found in Appendix 1. The rectified light is then given by

$$\ell_{rect} = \frac{\ell_{obs} - (B_1 \sin\theta + B_2 \sin 2\theta)}{\frac{1}{2} A_o + A_1 \cos\theta + A_2 \cos 2\theta} \qquad (2.25)$$

and the rectified phase by

$$\sin\theta_{rect} = \frac{\sin\theta_{obs}}{(1 - z\cos^2\theta_{obs})^{\frac{1}{2}}}$$

and

$$\cos\theta_{rect} = \left[\frac{1 - z}{1 - z\cos^2\theta_{obs}}\right]^{\frac{1}{2}} \cos\theta_{obs} \quad , \qquad (2.26)$$

where both equations are required for proper quadrant definition (in taking an inverse tangent, an electronic computer uses the range $-\pi/2 \leq \theta \leq \pi/2$, rather than $0 \leq \theta \leq 2\pi$, which is the range of θ_{obs}). These formulae apply to all observations, both in and out of eclipse. The factor z may be obtained empirically by using the approximate relation $z \simeq |2A_2|$. It should also be noted that the presence of the sine terms in the Fourier series for $\ell(\theta)$ is not justifiable physically; their only purpose is to take care of any extra 'complications' that might arise. This then is the process of rectification as developed by Russell and Merrill. The end product is a light curve that is flat outside the eclipses, with the eclipses being those appropriate to spherical stars.

The process of rectification is open to criticism on several grounds. The first, and most obvious, is the presence of the sine terms in the Fourier series for $\ell(\theta)$. The presence of such terms should be justifiable from a physical point of view, but the present author knows of no such justification, published or unpublished. Another criticism, raised by Kopal (1979, pg. 192), is that one is using a Fourier series outside its range of validity, since a series, which has been fitted to the out-of-eclipse observations is being applied to all observations, both in- and out-of-eclipse. Rectification will work for systems in which distortion effects and the reflection effects are minimal. The example in Appendix 1 is of this variety. In cases such as these, the B_n-terms are quite small in comparison to the A_n-terms. However, one really cannot apply rectification to highly distorted systems (very close binaries, e.g. W Ursae Majoris). In systems such as these, the shapes of the stars depart greatly from an ellipsoidal form, and actually approach a Roche-surface form. The theory upon which rectification rests is clearly not designed with such systems in mind. Consequently, rectification is no longer used, and more physically acceptable procedures have replaced it.

2.6 Differential Corrections

If the geometric elements r_1, r_2, i, L_1, L_2, x_1, and x_2 are well-determined (in the sense that the solution for these elements is determinate), one may improve the values of these elements by the 'differential corrections' procedure. Differential corrections are based on the idea that if

$$\ell(\theta) = u - \alpha L \qquad (u = \ell(90^{\circ})) \qquad (2.27)$$

then

$$\Delta\ell(\theta) = \Delta u - \alpha\Delta L - L\Delta\alpha$$

$$= \Delta u - \alpha\Delta L - L \sum_{j=1}^{N} \frac{\partial\alpha}{\partial x_j} \Delta x_j \quad , \qquad (2.28)$$

where x_j is one of the elements r_1, r_2, i, x_1, or x_2. Equation (2.28) may now be regarded as an equation of condition, so that if this equation is written out for each (θ,ℓ) pair, one may solve the system of equations for Δx_j's, Δu, and ΔL by the least-squares method. The value of $\Delta\ell$ is found by subtracting the calculated value of ℓ from the observed value (i.e., an '0-c'). The various partial derivatives appearing in equation (2.28) have different forms according to the eclipse type. The form of α must also be chosen according to the eclipse type. The paper by Irwin (1947) describes the procedure of differential corrections in great detail, and tables of the various derivatives are provided in an appendix to the paper. The present author has written a number of computer programs for performing the differential corrections procedure using the values of the derivatives from Irwin's tables or values generated by the equations for the derivatives. Some of these programs may be found in Appendix 1. It should be noted that one cannot apply differential corrections to every eclipsing binary star, since, as mentioned earlier, a well-determined set of elements is required, as well as a large number of observations to make the least-squares method truly applicable. Least-squares differential corrections should not be regarded as a 'black box' that always generates improved values of the elements, therefore it should be applied with some discretion. As Irwin mentions in the paper quoted earlier, least-squares is no substitute for good sense!

2.7 Conclusion

The discussion of the Russell model and the method of light curve analysis associated with it is now complete. This model of an eclipsing binary star is best applied to systems having spherical components, since any application to systems having distorted stars will inevitably lead to the use of rectification, the validity of which is in some doubt. The Russell-Merrill method is still used to provide a preliminary set of elements to be used in more advanced methods of light curve analysis. In short, the Russell-Merrill method is not the most fruitful one, since it is possible to derive much more information from a light curve. It would also be of some advantage to have a method of analysis tailored for use on an electronic computer.