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Position Estimation	Elliptical Confidence Region
Direction Finding	TI-59 Calculator
Bearings Only Fixing	

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

The Report provides a procedure for estimating an object's position based on bearings taken from or on the object for two or more stations. The Report also provides a program for the TI-59 calculator to implement the procedure.

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In this revision, the TI-59 program has been modified so that a user may now revise a location estimate by entering additional bearing data.

An example of the use of this option is given on Page 16.

TABLE OF CONTENTS

The programs in this report are for use within the Department of the Navy, and they are presented without representation or warranty of any kind.

A PROCEDURE FOR ESTIMATING AN OBJECT'S POSITION BASED ON TWO OR MORE BEARINGS WITH A PROGRAM FOR A TI-59 CALCULATOR

I. Introduction

A procedure for estimating an object's position with bearings taken on or from two or more stations is developed in Section IV of this report. In the development of the procedure, the following things are assumed: The object and the stations are fixed on the surface of a flat earth and the position of each station is known. The error in the bearing taken on or from a station is a normal random variable with a known standard deviation e and a mean of zero (if bias exists, it is known and removed); and station bearing errors are independent. The user instructions for a TI-59 program to implement the procedure are given in Section II, and the program listing is given in Section III.

As an example to illustrate a use of the program, suppose bearings are taken on an object from three stations (1, 2 and 3) as illustrated in Figure 1. Also, suppose that the assumptions stated above are satisfied and that an initial estimate of the object's position is made and that it is relatively near the object. This assumption is discussed in Section IV.

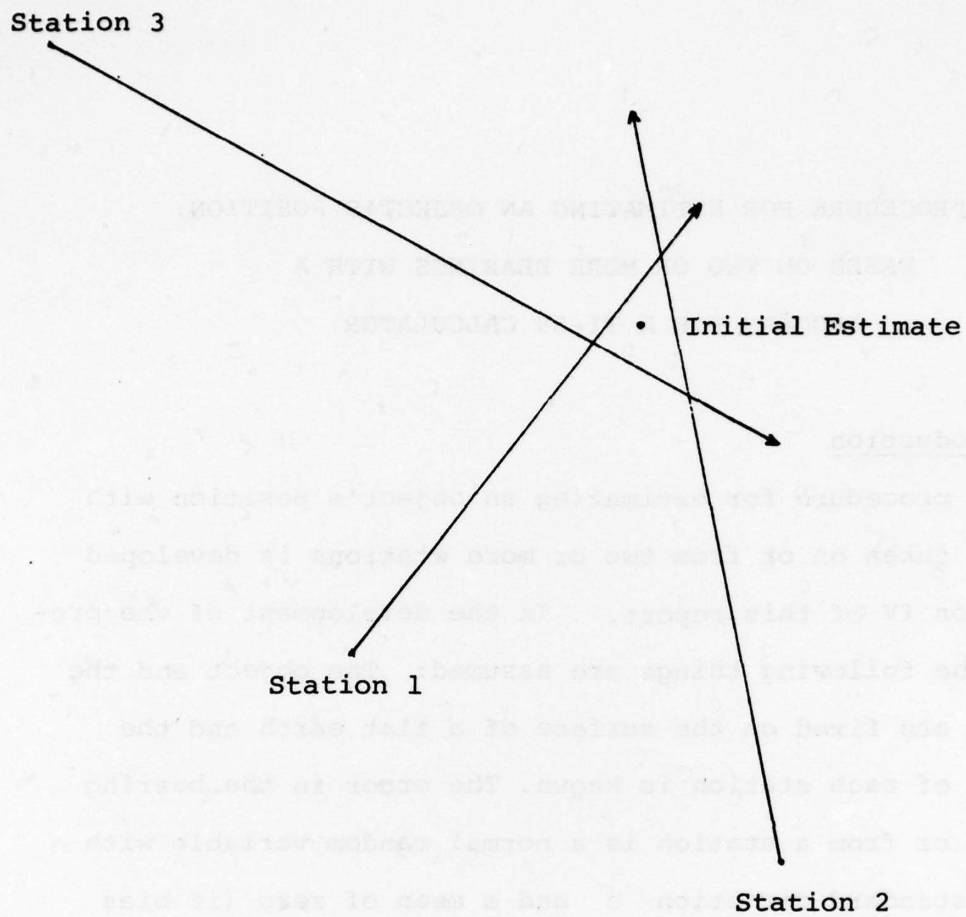


FIGURE 1. Geometry for the Example

Let the measured bearings and bearing errors (standard deviations) be:

$\theta_1 = 35°$ $\qquad e_1 = 4°$

$\theta_2 = 351°$ $\qquad e_2 = 7°$

$\theta_3 = 131°$ $\qquad e_3 = 5°$

And let the ranges and bearings of the initial estimate be:

r_1 = 10,000 meters, β_1 = 38°

r_2 = 15,000 meters, β_2 = 346°

r_3 = 12,000 meters, β_3 = 127° .

Use of the position estimation program with this data gives a final position estimate (fix) determined by:

$$x = -512 \text{ meters}$$

$$y = -75 \text{ meters}$$

where x is its East-West distance and y is its North-South distance from the initial position estimate. The East-West, North-South xy-coordinate system with its origin at the initial estimate is shown in Figure 2. So the final position estimate is 512 meters to the West and 75 meters to the South of the initial position estimate.

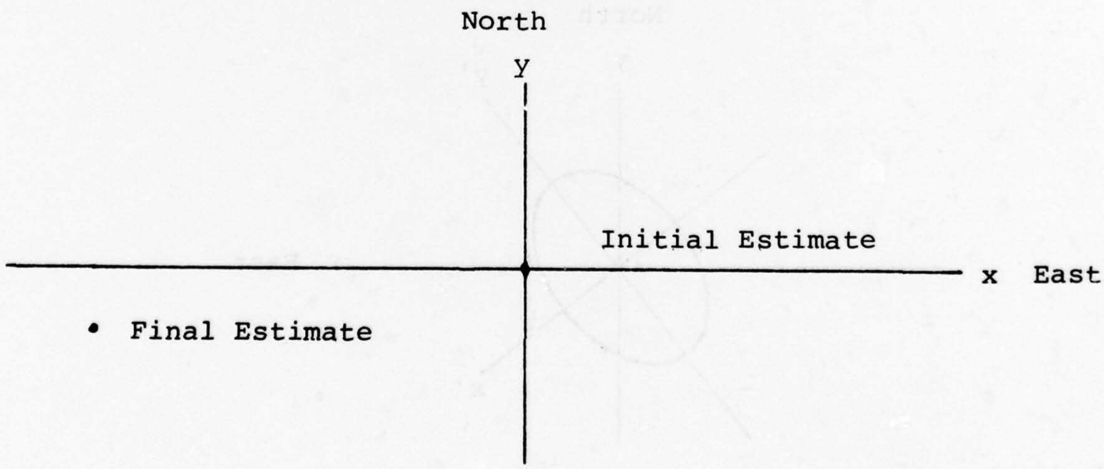


FIGURE 2. The Location of the Final Position Estimate with Respect to the Initial Position Estimate.

Minimum area elliptical confidence regions for an object's position can also be found by using the TI-59 program. The centers of the regions are at the fix, and their axes lie along the x' and y' axes of the coordinate system obtained by rotating the East-West, North-South xy-coordinate system with origin at the fix through an angle γ. The angle γ is defined so that it is positive for a rotation in the counterclockwise direction.

With the data from the above example, the program gives $\gamma = -31°$; so, the x' axis is directed 31° South of East. For a confidence region with minimum area and a confidence level of .9000, the ellipse bounding the region has a semi-major axis of 2064 meters, and a semi-minor axis of 1453 meters. The area of the region is 9.43 square killometers or 2.75 square nautical miles. The region is shown in Figure 3.

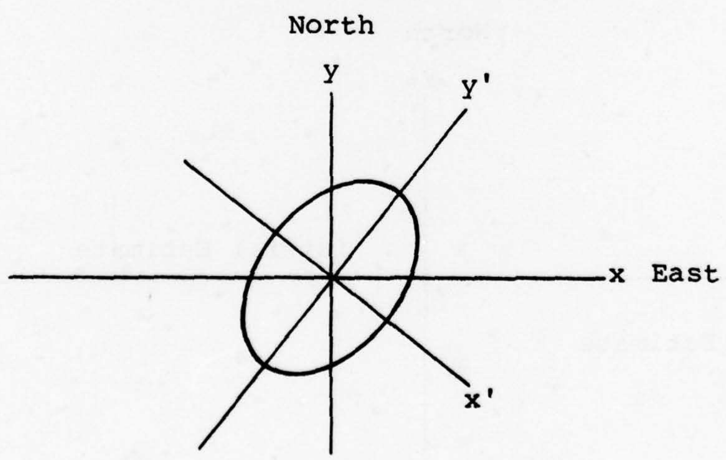


FIGURE 3. A .9000 Confidence Region for an Object's Position.

In the example discussed above, the position of the initial estimate is an input to the program. If this is not desirable, the program can be used to determine a position for the initial estimate. The position is the intersection of the two bearing lines corresponding to the first two bearings entered in the program. Both options are illustrated in Section II.

Since, in general, the smaller the bearing errors, the more likely that the initial estimate will be relatively near the object; small bearing errors can be considered to be a condition on the use of the procedure.

Note, if the length of the base line joining the first two stations is small enough and their bearing errors are large enough, observed bearing lines from the two stations may not intersect. If they do not intersect, the initial estimate determined by the program will be at the intersection of the reciprocal bearing lines, and a gross error can result.

Encl (2)

II. User Instructions

The TI-59 program to which the user instructions in this section apply can be used to calculate the quantities described in Section I.

The program requires the following inputs:

1. the observed bearing from or on an object for two or more stations;
2. station positions relative to a reference position; and
3. the bearing error (standard deviation) for each observed bearing.

Station positions can be specified in either of two ways. In the first way, Mode A, each station's position is specified in terms of its bearing α and its range ρ from a reference position. In the second way, Mode B, each station's position is specified in terms of its East-West distance x (plus for East) and its North-South distance (plus for North) from a reference position. The reference position can be any convenient location. For example, if it were at a station, then for that station $\alpha = 0$ and $\rho = 0$ or $x = 0$ and $y = 0$.

The program also requires an initial estimate of the object's position. The user has two options:

1. Let the program provide an estimate, or
2. Provide one with the input data.

For Option 1, the initial estimate is at the intersection of the bearing lines determined by the first two observed bearings entered into the program. For this reason, if this option is chosen, the first and second groups of data entered should correspond to the two stations estimated to have the smallest products $r_i e_i$. Although in this option the reference position cannot be at the initial estimate, it can be at one of the stations. If only two stations are involved, the final estimate is at the intersections of the bearing lines. (If the second option of either mode is used with an initial estimate which is not at the intersection of the two bearing lines, the coordinates of the final estimate will differ from coordinates of the intersection to the degree of the approximations involved in the estimation procedure.)

Two ways of providing confidence (probability) region data are available. In the first way, Mode C, the confidence (probability) p is specified. In the second way, Mode D, the multiplier k is specified where $k\sigma_{\hat{x}}$, and $k\sigma_{\hat{y}}$, are the semi-axes of the bounding ellipse.

The values of various quantities calculated by the program are either stored in registers or appear in the display. If a PC-100A printer is used, some of these values will be printed. The location of calculated values and the printing format is given after the user instructions. Those quantities which are not described in Section I are described below in the User Instructions or in Section IV.

All angles required or calculated by the program are in decimal degrees.

Step	Instructions	Enter	Press	Display
1.	If the calculator has been in use and flags have been set or the memory repartitioned, turn the calculator off and then on.			
2.	Read Side 1 and Side 2 of Card 1.			
3.	Read Side 3 of Card 2.			
MODE A:	Station Locations Specified in Terms of Bearing and Range from a Reference Point.			
4a.	If the initial position estimate will be determined by the program, go to Step 7a. See the note on Page 10.			
5a.	Enter the initial estimate's bearing.	α^*	A'	
6a.	Enter the initial estimate's range.	ρ^*	R/S	
7a.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	A	
8a.	Enter the station's bearing.	α_i	R/S	
9a.	Enter the station's range.	ρ_i	R/S	
10a.	Enter the bearing error.	e_i	R/S	i
11a.	Repeat Steps 7a, 8a, 9a and 10a for all stations. The number of repetitions i appears in the display after Step 10a.			

Step	Instructions	Enter	Press	Display
MODE B:	Station Locations Specified in Terms of East-West Distance and North-South Distance from a Reference Point.			
4b.	If the initial position estimate will be determined by the program, go to Step 7b. See the note on Page 10.			
5b.	Enter the initial estimate's East-West distance.	x^*	B'	
6b.	Enter the initial estimate's North-South distance.	y^*	R/S	
7b.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	B	
8b.	Enter the station's East-West distance.	x_i	R/S	
9b.	Enter the station's North-South distance.	y_i	R/S	
10b.	Enter the bearing error.	e_i	R/S	i
11b.	Repeat Steps 7b, 8b, 9b and 10b for all stations. The number of repetitions i appears in the display after Step 10b.			
BOTH MODES				
12.	Calculate the East-West distance, the North-South distance, the bearing and the range of the position estimate relative to the reference position. Also calculate the rotation angle γ, and the standard deviations $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$. To include additional bearing measurements after this calculation, go to Step 18.		R/S	

Step	Instructions	Enter	Press	Display
13.	For confidence (probability) region calculations, go to Step 14 if the confidence (probability) for the region is specified. If k is specified where $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ are the semi-axes of the bounding ellipse with the larger the major axis, go to Step 16.			
14.	Enter p, the confidence level (probability) and calculate k, $k\sigma_{\hat{x}'}$, $k\sigma_{\hat{y}'}$ and the area of the region. (The area units correspond to the distance units used.)	p	C	Area
15.	For a different value of p, go to Step 14.			
16.	Enter k and calculate the confidence level (probability) p, $k\sigma_{\hat{x}'}$, $k\sigma_{\hat{y}'}$ and the area of the region. (The area units correspond to the distance units used.)	k	D	Area
17.	For a different value of k, go to Step 16.			
18.	To include an additional bearing measurement from either a new or old station, go to Step 7a if using Mode A or Step 7b if using Mode B.			

NOTE: If a data entry error occurs in either mode, press RST and then use the following procedure: For Option 1, return to Step 7 and repeat all data entries. For Option 2, return to Step 5 and repeat all data entries.

Also, if a position estimate is to be determined for a new object position or if a new mode is to be used, follow this instruction.

NOTES:

a) The program printing format is given below:

For the initial data, i = 1, 2, ... , n with one space between groups:

Mode A		Mode B
α^*	initial estimate if provided	x^*
ρ^*		y^*
θ_i		θ_i
α_i		x_i
ρ_i		y_i
e_i		e_i

The format for the calculated position data is:

x

y

α

ρ

γ

$\sigma_{\hat{x}'}$

$\sigma_{\hat{y}'}$

For the confidence (probability) region portion of the program the format is after pressing either C or D:

p	
k	
$k\sigma_{\hat{x}'}$	semi-axis
$k\sigma_{\hat{y}'}$	semi-axis
Area	

b) The following data is stored in the indicated registers:

Data	Registers
x*	R38
y*	R39
γ	R29
x	R30
y	R31
α	R32
ρ	R33
$\sigma_{\hat{x}'}$	R16
$\sigma_{\hat{y}'}$	R17
p	R14
k	R15
$k\sigma_{\hat{x}'}$	R18
$k\sigma_{\hat{y}'}$	R19

Four data tapes for a sample problem are given below. Distance units have not been specified, but they could be meters for example. Angles are in degrees. Option 1 (initial estimate not provided) for Mode A and Mode B is indicated by A and by B and Option 2 (initial estimate provided) is indicated by A' and B'.

For each mode and each option, the input data are indicated. The data determine the relative locations of three stations as well as the observed bearing of an object from each station.

For A and B, the reference location is at Station 1 and the initial position estimate (determined by the program) is at the intersection of the bearing lines for Station 1 and Station 2.

The intersection has coordinates $x^* = 906.4853528$ and $y^* = 17296.77092$ with respect to Station 1.

For A' and B', both the initial estimate and the reference location are at the intersection of the bearing lines, so $\alpha^* = 0$ and $\rho^* = 0$ and $x^* = 0$ and $y^* = 0$.

The data for A, B, A' and B' are all equivalent, and each solution gives the same data for a confidence (probability) region calculation. A tape with confidence (probability) region results for both Mode C and Mode D which correspond to A, B, A' and B' is given with the first four data tapes.

A	
3.	θ_1
0.	α_1
0.	ρ_1
4.	e_1
33.	θ_2
273.	α_2
10000.	ρ_2
3.	e_2
303.	θ_3
33.	α_3
14000.	ρ_3
8.	e_3
573.5878933	x
16462.71223	y
1.995471725	α
16472.70157	ρ
-7.325392245	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

A'	
0.	α^*
0.	ρ^*
3.	θ_1
183.	α_1
17320.50808	ρ_1
4.	e_1
33.	θ_2
213.	α_2
20000.	ρ_2
3.	e_2
303.	θ_3
129.5867755	α_3
8717.797886	ρ_3
8.	e_3
-332.8974567	x
-834.0586835	y
201.7584019	α
898.0393111	ρ
-7.325392259	γ
787.3663757	$\sigma_{\hat{x}'}$
1233.080776	$\sigma_{\hat{y}'}$

B

3.	θ_1
0.	x_1
0.	y_1
4.	e_1
33.	θ_2
-9986.295348	x_2
523.3595624	y_2
3.	e_2
303.	θ_3
7624.94649	x_3
11741.38795	y_3
8.	e_3
573.5878927	x
16462.71223	y
1.995471723	α
16472.70157	ρ
-7.325392245	γ
787.3663756	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

B'

0.	x^*
0.	y^*
3.	θ_1
-906.4853528	x_1
-17296.77092	y_1
4.	e_1
33.	θ_2
-10892.7807	x_2
-16773.41136	y_2
3.	e_2
303.	θ_3
6718.461137	x_3
-5555.382969	y_3
8.	e_3
-332.8974589	x
-834.0586906	y
201.7584019	α
898.0393184	ρ
-7.325392239	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

C or D

0.9	p
2.145966026	k
1689.661493	$k\sigma_{\hat{x}'}$
2646.149454	$k\sigma_{\hat{y}'}$
14046364.97	Area
.8646647168	p
2.	k
1574.732751	$k\sigma_{\hat{x}'}$
2466.161553	$k\sigma_{\hat{y}'}$
12200517.6	Area

The following data tape illustrates the effects of using only the bearings for the first two stations. The tape is for Mode A, Option 1. The values stored in Registers 38 and 39 (the x and y coordinates of the intersection of the bearing lines from Station 1 and Station 2 with reference, in this case, to Station 1) are also listed on the tape (as well as given above). And, as can be seen, the initial estimate and final estimate correspond.

The data tape also illustrates the use of additional bearing data to revise a position estimate. The data for Station 3 printed after the first confidence (probability) region calculation results was entered by again repeating Step 11a, and the remaining results were obtained by next repeating Step 12 and then Steps 14 and 16. Note, these results are the same as the corresponding results for Mode A on page 14.

A

3.	θ_1
0.	α_1
0.	ρ_1
4.	e_1

33.	θ_2
273.	α_2
10000.	ρ_2
3.	e_2

906.4853528	x
17296.77092	y
3.	α
17320.50808	ρ

-20.35750198	γ
818.886822	$\sigma_{\hat{x}'}$
3092.663848	$\sigma_{\hat{y}'}$

0.9	p
2.145966026	k
1757.303299	$k\sigma_{\hat{x}'}$
6636.751549	$k\sigma_{\hat{y}'}$
36639720.91	Area

.8646647168	p
2.	k
1637.773644	$k\sigma_{\hat{x}'}$
6185.327696	$k\sigma_{\hat{y}'}$
31824857.22	Area

303.	θ_3
33.	α_3
14000.	ρ_3
8.	e_3

573.5878933	x
16462.71223	y
1.995471725	α
16472.70157	ρ

-7.325392245	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

0.9	p
2.145966026	k
1689.661492	$k\sigma_{\hat{x}'}$
2646.149455	$k\sigma_{\hat{y}'}$
14046364.98	Area

.8646647168	p
2.	k
1574.732751	$k\sigma_{\hat{x}'}$
2466.161554	$k\sigma_{\hat{y}'}$
12200517.6	Area

906.4853528	R38
17296.77092	R39

To obtain the results given in Section I, use A' and take the reference position at the initial estimate ($\alpha^* = 0$, $\rho^* = 0$). Then $\alpha_1 = 218°$, $\alpha_2 = 166°$ and $\alpha_3 = 307°$. The data tape for the calculation is given below.

A'

0.	α^*
0.	ρ^*
35.	θ_1
218.	α_1
10000.	ρ_1
4.	e_1
351.	θ_2
166.	α_2
15000.	ρ_2
7.	e_2
131.	θ_3
307.	α_3
12000.	ρ_3
5.	e_3
-511.961856	x
-75.43753883	y
261.617789	α
517.4898687	ρ
-31.23492683	γ
677.2632305	$\sigma_{\hat{x}'}$
961.6888632	$\sigma_{\hat{y}'}$
0.9	p
2.145966026	k
1453.383883	$k\sigma_{\hat{x}'}$
2063.751628	$k\sigma_{\hat{y}'}$
9422966.381	Area
.8646647168	p
2.	k
1354.526461	$k\sigma_{\hat{x}'}$
1923.377726	$k\sigma_{\hat{y}'}$
8184684.605	Area

III. Program Listing

Before entering the program, press 2nd and then CP or turn the calculator off and then on. Next enter 5 in the display, press 2nd and then Op 17. This repartitions the calculator's memory so that the complete program can be entered.

Before recording the program, enter 6 in the display, press 2nd and then Op 17. This returns the calculator's memory to the normal partition (479.59). Returning the calculator to the normal partition allows the two program cards to be read in the normal partition without forcing. When the program is used, it repartitions the calculator so that Bank 3 registers are program registers.

```
000  76  LBL
001  15   E
002  09   9
003  42  STO
004  00   00
005  42  STO
006  01   01
007  92  RTN
008  76  LBL
009  18  C'
010  69  OP
011  20   20
012  72  ST*
013  00   00
014  92  RTN
015  76  LBL
016  19  D'
017  69  OP
018  21   21
019  73  RC*
020  01   01
021  92  RTN
022  76  LBL
023  10  E'
024  65   ×
025  89   π
026  55   ÷
027  01   1
028  08   8
029  00   0
030  95   =
031  92  RTN
032  76  LBL
033  12   B
034  86  STF
035  01   01
036  76  LBL
037  11   A
038  32  X:T
039  87  IFF
040  00   00
041  00   00
042  59   59
043  87  IFF
044  03   03
045  00   00
046  59   59
047  05   5
048  69  OP
049  17   17
050  47  CMS
051  15   E
052  02   2
053  42  STO
054  09   09
055  69  OP
056  28   28
057  86  STF
058  00   00
059  32  X:T
060  99  PRT
061  18  C'
062  91  R/S
063  99  PRT
064  18  C'
065  91  R/S
066  99  PRT
067  18  C'
068  91  R/S
069  99  PRT
070  10  E'
071  18  C'
072  87  IFF
073  03   03
074  01   01
075  67   67
076  19  D'
077  87  IFF
078  01   01
079  00   00
080  91   91
081  75   -
082  19  D'
083  95   =
084  94  +/-
085  38  SIN
086  65   ×
087  19  D'
088  61  GTO
089  01   01
090  02   02
091  85   +
092  19  D'
093  32  X:T
094  19  D'
095  22  INV
096  37  P/R
097  24  CE
098  95   =
099  39  COS
100  65   ×
101  32  X:T
102  95   =
103  48  EXC
104  19   19
105  22  INV
106  97  DSZ
107  09   09
108  01   01
109  15   15
110  01   1
111  69  OP
112  21   21
113  98  ADV
114  91  R/S
115  42  STO
116  18   18
117  65   ×
118  43  RCL
119  14   14
120  42  STO
121  20   20
122  38  SIN
123  75   -
124  43  RCL
125  10   10
126  22  INV
127  44  SUM
128  20   20
129  38  SIN
130  65   ×
131  43  RCL
132  19   19
133  95   =
134  55   ÷
135  43  RCL
136  20   20
137  38  SIN
138  95   =
139  42  STO
140  38   38
141  43  RCL
142  14   14
143  39  COS
144  65   ×
145  43  RCL
146  18   18
147  75   -
148  43  RCL
149  10   10
```

```
150  39  COS
151  65   ×
152  43  RCL
153  19   19
154  95   =
155  55   ÷
156  43  RCL
157  20   20
158  38  SIN
159  95   =
160  42  STO
161  39   39
162  86  STF
163  03   03
164  02   2
165  42  STO
166  09   09
167  15   E
168  19  D'
169  10  E'
170  42  STO
171  18   18
172  19  D'
173  87  IFF
174  01   01
175  01   01
176  81   81
177  32  X:T
178  19  D'
179  32  X:T
180  37  P/R
181  75   -
182  43  RCL
183  38   38
184  95   =
185  94  +/-
186  32  X:T
187  22  INV
188  87  IFF
189  01   01
190  01   01
191  93   93
192  19  D'
193  75   -
194  43  RCL
195  39   39
196  95   =
197  94  +/-
198  32  X:T
199  22  INV
```

```
200  37  P/R
201  42  STO
202  19   19
203  10  E'
204  75   -
205  43  RCL
206  18   18
207  95   =
208  94  +/-
209  32  X:T
210  69  OP
211  21   21
212  64  PD*
213  01   01
214  42  STO
215  18   18
216  89   π
217  32  X:T
218  22  INV
219  77   GE
220  02   02
221  26   26
222  75   -
223  32  X:T
224  65   ×
225  02   2
226  95   =
227  49  PRD
228  18   18
229  73  RC*
230  01   01
231  35  1/X
232  42  STO
233  27   27
234  42  STO
235  26   26
236  32  X:T
237  43  RCL
238  19   19
239  37  P/R
240  42  STO
241  28   28
242  49  PRD
243  27   27
244  33  X²
245  44  SUM
246  23   23
247  43  RCL
248  18   18
249  49  PRD
```

```
250  26   26
251  49  PRD
252  27   27
253  32  X:T
254  49  PRD
255  28   28
256  49  PRD
257  26   26
258  33  X²
259  44  SUM
260  21   21
261  43  RCL
262  28   28
263  44  SUM
264  22   22
265  43  RCL
266  26   26
267  44  SUM
268  24   24
269  43  RCL
270  27   27
271  44  SUM
272  25   25
273  22  INV
274  97  DSZ
275  09   09
276  02   02
277  81   81
278  61  GTO
279  01   01
280  68   68
281  69  OP
282  28   28
283  43  RCL
284  08   08
285  98  ADV
286  91  R/S
287  43  RCL
288  21   21
289  42  STO
290  41   41
291  43  RCL
292  22   22
293  42  STO
294  42   42
295  43  RCL
296  23   23
297  42  STO
298  43   43
299  43  RCL
```

300	23	23
301	42	STO
302	10	10
303	65	×
304	43	RCL
305	21	21
306	22	INV
307	44	SUM
308	10	10
309	75	-
310	43	RCL
311	22	22
312	22	INV
313	49	PRD
314	10	10
315	33	X^2
316	95	=
317	35	1/X
318	49	PRD
319	41	41
320	49	PRD
321	42	42
322	49	PRD
323	43	43
324	43	RCL
325	10	10
326	35	1/X
327	65	×
328	02	2
329	95	=
330	22	INV
331	30	TAN
332	55	÷
333	02	2
334	95	=
335	42	STO
336	29	29
337	43	RCL
338	43	43
339	65	×
340	43	RCL
341	24	24
342	75	-
343	43	RCL
344	42	42
345	65	×
346	43	RCL
347	25	25
348	85	+
349	43	RCL
350	38	38
351	95	=
352	42	STO
353	30	30
354	99	PRT
355	32	X:T
356	43	RCL
357	42	42
358	65	×
359	43	RCL
360	24	24
361	75	-
362	43	RCL
363	41	41
364	65	×
365	43	RCL
366	25	25
367	85	+
368	43	RCL
369	39	39
370	95	=
371	42	STO
372	31	31
373	99	PRT
374	32	X:T
375	22	INV
376	37	P/R
377	99	PRT
378	42	STO
379	32	32
380	01	1
381	32	X:T
382	99	PRT
383	42	STO
384	33	33
385	98	ADV
386	43	RCL
387	29	29
388	99	PRT
389	37	P/R
390	42	STO
391	11	11
392	33	X^2
393	42	STO
394	12	12
395	42	STO
396	13	13
397	32	X:T
398	49	PRD
399	11	11
400	33	X^2
401	42	STO
402	14	14
403	42	STO
404	15	15
405	43	RCL
406	41	41
407	49	PRD
408	12	12
409	49	PRD
410	14	14
411	43	RCL
412	42	42
413	65	×
414	02	2
415	95	=
416	49	PRD
417	11	11
418	43	RCL
419	43	43
420	49	PRD
421	13	13
422	49	PRD
423	15	15
424	43	RCL
425	15	15
426	85	+
427	43	RCL
428	11	11
429	85	+
430	43	RCL
431	12	12
432	95	=
433	34	√X
434	99	PRT
435	42	STO
436	16	16
437	43	RCL
438	13	13
439	75	-
440	43	RCL
441	11	11
442	85	+
443	43	RCL
444	14	14
445	95	=
446	34	√X
447	99	PRT
448	42	STO
449	17	17

450	98	ADV
451	91	R/S
452	76	LBL
453	17	B'
454	86	STF
455	02	02
456	76	LBL
457	16	A'
458	99	PRT
459	32	X:T
460	05	5
461	69	OP
462	17	17
463	47	CMS
464	15	E
465	91	R/S
466	99	PRT
467	87	IFF
468	02	02
469	04	04
470	73	73
471	32	X:T
472	37	P/R
473	42	STO
474	38	38
475	32	X:T
476	42	STO
477	39	39
478	86	STF
479	03	03
480	98	ADV
481	91	R/S
482	76	LBL
483	14	D
484	86	STF
485	04	04
486	76	LBL
487	13	C
488	98	ADV
489	87	IFF
490	04	04
491	05	05
492	36	36
493	42	STO
494	14	14
495	99	PRT
496	75	-
497	01	1
498	95	=
499	94	+/-
500	23	LNX
501	65	×
502	02	2
503	95	=
504	94	+/-
505	34	√X
506	42	STO
507	15	15
508	99	PRT
509	65	×
510	43	RCL
511	16	16
512	95	=
513	99	PRT
514	42	STO
515	18	18
516	65	×
517	53	(
518	43	RCL
519	15	15
520	65	×
521	43	RCL
522	17	17
523	54	)
524	99	PRT
525	42	STO
526	19	19
527	65	×
528	89	π
529	95	=
530	99	PRT
531	98	ADV
532	22	INV
533	86	STF
534	04	04
535	91	R/S
536	42	STO
537	15	15
538	33	X^2
539	55	÷
540	02	2
541	95	=
542	94	+/-
543	22	INV
544	23	LNX
545	75	-
546	01	1
547	95	=
548	94	+/-
549	42	STO
550	14	14
551	99	PRT
552	43	RCL
553	15	15
554	61	GTO
555	05	05
556	08	08
557	00	0
558	00	0
559	00	0

IV. A Development for the Procedure

In the development for the estimation procedure given here, all angles are in radians and the assumptions stated in Section I apply.

Figure 4 shows three bearing lines from the ith of n stations. One is the observed bearing line of an object. One of length $\imath_i$ goes to the origin of an xy-coordinate system located at the object's unknown position. And one of length r_i goes to an initial estimate with known position but unknown coordinates (x,y). Note, estimates for $-x$ and $-y$ estimate the object's position. To find estimates $-\hat{x}$ and $-\hat{y}$, consider the arc coordinates $u_i = \imath_i(\theta_i-\phi_i)$ of the observed bearing line and $v_i = \imath_i(\beta_i-\phi_i)$ of the bearing line to the point (x,y). They are defined by the three bearing lines and the circle of radius $\imath_i$ which goes through the object's position and which is centered on the station as shown in Figure 4.

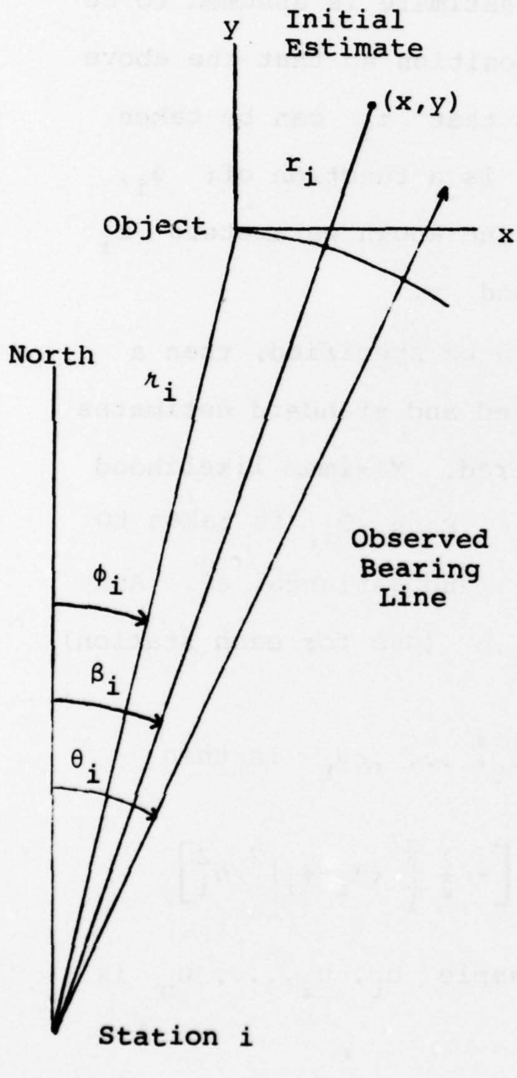


FIGURE 4. Problem Geometry.

By defining $w_i = r_i(\theta_i - \beta_i)$ (all angles in radians), $u_i = v_i + w_i$. Note, $\theta_i - \beta_i$ is known, but $\beta_i - \phi_i$ is not. However, v_i can be expressed in terms of x and y, and, to first order, $v_i = x \cos \beta_i - y \sin \beta_i$; so, if $\tan(\beta_i - \phi_i) \simeq (\beta_i - \phi_i)$ for $i = 1,2,\ldots,n$, that is, if (x,y) is relatively near the object's position, $u_i \simeq r_i(\theta_i - \beta_i) + x \cos \beta_i - y \sin \beta_i$ for $i = 1,2,\ldots,n$.

In this development, the initial estimate is assumed to be relatively close enough to the object's position so that the above approximation for u_i can be used and so that r_i can be taken equal to r_i. With this assumption, u_i is a function of: θ_i, the observed value of a random quantity; the known parameters r_i and β_i; and the unknown parameters x and y.

If a distribution for the Θ_i can be specified, then a distribution for the U_i can be determined and standard estimates $\hat{x}$ and $\hat{y}$ for x and y can be considered. Maximum likelihood estimates are discussed in this section. Each Θ_i is taken to be a normal random variable with mean ϕ_i and variance e_i^2. And the n random variables Θ_i, $i = 1,2,\ldots,n$ (one for each station) are taken to be independent.

The likelihood for a sample $\theta_1, \theta_2, \ldots, \theta_n$ is then

$$L(\theta_1,\theta_2,\ldots,\theta_n) = \prod_1^n \left(\frac{1}{\sqrt{2\pi}\, e_i}\right) \exp\left[-\frac{1}{2}\sum_1^n (\theta_i - \phi_i)^2/e_i^2\right]$$

and the likelihood for a corresponding sample $u_1, u_2, \ldots, u_n$ is

$$L(u_1,u_2,\ldots,u_n) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma_i}\right) \exp\left[-\frac{1}{2}\sum_1^n u_i^2/\sigma_i^2\right]$$

where $\sigma_i = r_i e_i$ (with e_i in radians) since $u_i = r_i(\theta_i - \phi_i)$.

By definition, the maximum likelihood estimates of x and y are the estimates $\hat{x}$ and $\hat{y}$ which make $L(u_1,u_2,\ldots,u_n)$ a maximum. In this case, making $L(u_1,u_2,\ldots,u_n)$ a maximum is equivalent to making $\sum_1^n(u_i^2/\sigma_i^2)$ a minimum. So, to find $\hat{x}$ and $\hat{y}$, solve the following two equations for x and y:

$$\frac{\partial(\ell n\ L)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial(\ell n\ L)}{\partial y} = 0\ .$$

The solutions are $x = \hat{x}$ and $y = \hat{y}$, and $\hat{x}$ and $\hat{y}$ are the maximum likelihood estimates. With $w_i = r_i(\theta_i - \beta_i)$ and the conditions assumed above these two equations are linear equations in x and y. And,

$$\sum_1^n [w_i + \hat{x}\cos\beta_i - \hat{y}\sin\beta_i](\cos\beta_i)/\sigma_i^2 = 0$$

and

$$\sum_1^n [w_i + \hat{x}\cos\beta_i - \hat{y}\sin\beta_i](\sin\beta_i)/\sigma_i^2 = 0\ .$$

And, in terms of the following quantities:

$$A = \Sigma(\cos^2\beta_i)/\sigma_i^2\ , \qquad B = \Sigma(\sin\beta_i\cos\beta_i)/\sigma_i^2\ ,$$

$$C = \Sigma(\sin^2\beta_i)/\sigma_i^2\ , \qquad D = \Sigma(w_i\cos\beta_i)/\sigma_i^2\ ,$$

$$E = \Sigma(w_i\sin\beta_i)/\sigma_i^2\ ,$$

the equations are:

$$A\hat{x} - B\hat{y} = -D$$

$$B\hat{x} - C\hat{y} = -E \ .$$

So the solutions are:

$$\hat{x} = (BE - CD)/(AC - B^2)$$

$$\hat{y} = (AE - BD)/(AC - B^2) \ .$$

A confidence region can be constructed about an estimated position. In order to indicate how this is done, a probability region about the true position will be considered first.

Note, $\hat{x}$ and $\hat{y}$ are values of random variables. If a new set of bearings $\theta_1, \theta_2, \ldots, \theta_n$ is observed (for a fixed initial estimate and object), in general, a new pair of values $\hat{x}$ and $\hat{y}$ will be obtained.

If $\hat{X}$ and $\hat{Y}$ represent these random variables, then

$$\hat{X} = \frac{1}{(AC-B^2)} \sum_1^n (W_i/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)$$

$$\hat{Y} = \frac{1}{(AC-B^2)} \sum_1^n (W_i/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)$$

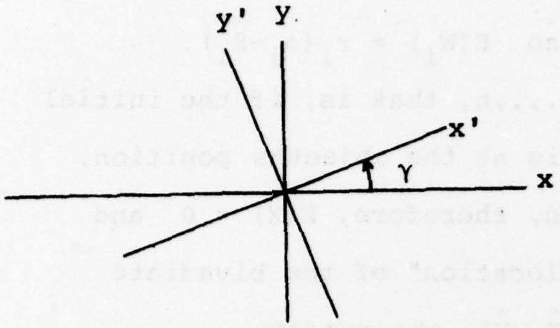
with $W_i = r_i(\Theta_i - \beta_i)$. ($W_i$ is the random distance intercepted along the ith arc between the bearing lines defined by Θ_i and β_i.)

Note, $\hat{X}$ and $\hat{Y}$ have a bivariate normal distribution, since they are a linear combination of the n normal random variables $W_1, W_2, \ldots, W_n$, or equivalently of the n normal random variables $\Theta_1, \Theta_2, \ldots, \Theta_n$. Also $E(W_i) = r_i(\phi_i - \beta_i)$.

If $\beta_i = \phi_i$ for $i = 1,2,\ldots,n$, that is, if the initial estimate of the object's position is at the object's position, $E(W_1) = 0$ for $i = 1,2,\ldots,n$. And, therefore, $E(\hat{X}) = 0$ and $E(\hat{Y}) = 0$. So, in this case, the "location" of the bivariate normal distribution of a point $(\hat{X}, \hat{Y})$, the random coordinates of the object's estimated position, is the same as that for the point $(-\hat{X}, -\hat{Y})$ and both are centered on the object's position. However, the "location" of the distribution of $(-\hat{X}, -\hat{Y})$ is independent of the location of the initial estimate when the coordinates $(-\hat{X}, -\hat{Y})$ refer to a coordinate system with origin at the initial estimate. This fact simplifies the establishment of a confidence region about the location of an estimated position.

A region of minimum area for a given probability of containment of an estimated position can be determined. The region is bounded by an ellipse which is centered on the object's position and whose axes lie along the axes of an x'y'-coordinate system obtained by rotating the xy-coordinate system centered on the object's position through an angle γ. In this system, $\sigma_{\hat{x}\hat{y}} = 0$, that is, $\hat{X}'$ and $\hat{Y}'$ are independent normal random variables.

The two coordinate systems are illustrated in Figure 5. The coordinates of a point in the two systems are related by



$$x' = x \cos\gamma + y \sin\gamma$$

$$y' = -x \sin\gamma + y \cos\gamma$$

These relations, along with $\sigma_{\hat{x}'\hat{y}'} = 0$, imply:

FIGURE 5. Rotation Geometry.

$$\sigma^2_{\hat{x}'} = \sigma^2_{\hat{x}} \cos^2\gamma + 2\sigma_{\hat{x}\hat{y}} \cos\gamma \sin\gamma + \sigma^2_{\hat{y}} \sin^2\gamma ,$$

$$\sigma^2_{\hat{y}'} = \sigma^2_{\hat{x}} \sin^2\gamma - 2\sigma_{\hat{x}\hat{y}} \cos\gamma \sin\gamma + \sigma^2_{\hat{y}} \cos^2\gamma$$

and

$$\tan 2\gamma = \frac{2\sigma_{\hat{x}\hat{y}}}{\sigma^2_{\hat{x}} - \sigma^2_{\hat{y}}}$$

where γ, the angle of rotation of the coordinate axes, is positive in the counterclockwise direction.

With the initial estimate of the object's position at the object's position $(\beta_i = \phi_i,\ i = 1,2,\ldots,n)$, so $E(W_i) = 0$ and $Var(W_i) = \sigma_i^2$,

$$\sigma^2_{\hat{x}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)^2,$$

$$\sigma^2_{\hat{y}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)^2$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)(A \sin \beta_i - B \cos \beta_i).$$

Using the definition for A, B and C, the above become

$$\sigma^2_{\hat{x}} = \frac{C}{(AC-B^2)},$$

$$\sigma^2_{\hat{y}} = \frac{A}{(AC-B^2)},$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{B}{(AC-B^2)}.$$

So, $\tan 2\gamma = 2B/(C-A)$ for $\beta_i = \phi_i$, $i = 1,2,\ldots,n$.

With the object's position known and, hence, ϕ_i known for $i = 1,2,\ldots,n$, the above equations for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be used, since the initial estimate of the object's position can be taken as the object's position.

With values for $\sigma_{\hat{x}}$, $\sigma_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ, values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be found by using the equations in the middle of Page 30. And then, the probability that an estimated position will be within an ellipse of semiaxes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$

which is centered on the object's position can be found. It is $1 - \exp(-k^2/2)$. (This result follows from integrating the bivariate normal density over the ellipse.) And the area of the ellipse is $\pi k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$.

Given estimates $\hat{x}$ and $\hat{y}$ found by using the relations on Page 28, an ellipse with semi-axes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ centered on the point with coordinates $(-\hat{x},-\hat{y})$ in a coordinate system with origin at the initial estimate and oriented as indicated by γ is a $1 - \exp(-k^2/2)$ confidence region. This follows from the bivariate normal distribution of $-\hat{X}$ and $-\hat{Y}$ which in this system is centered on the object's position. The ellipse is defined if $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$ and $\sigma_{\hat{x}\hat{y}}$ are known (the covariance matrix is known). And to the degree of the approximations involved, this can be assumed to be the case. In particular, by assuming the initial estimate of the object's position is at the object's position, which is consistent with assuming $(\beta_i-\phi_i)$ is small, values for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be obtained by using the relations on Page 31. These values can then be used to determine $\sigma^2_{\hat{x}'}$ and $\sigma^2_{\hat{y}'}$ by using the relations on Page 30. And, then, with a value for k, a confidence region can be constructed. To the degree of the approximations involved, the shape of the confidence region is independent of both the object's position and of the initial estimate of the object's position.

For the case where bearings are taken from the object on two or more stations, θ_i is the reciprocal of the bearing taken from the object.

A discussion for this and for other bearings only position estimation procedures for situations similar to the one considered here is given in Reference 1 listed below. Reference 2 gives an equivalent bearings only procedure. It also gives a range only procedure, a range and bearing procedure and HP-9830A programs with which to implement the procedures. Using the fix determined by two lines of bearing as the initial estimate was suggested by this reference.

The equations used in the program to determine (x^*, y^*), the coordinates of the fix, are:

$$x^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \sin\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \sin\theta_1$$

$$y^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \cos\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \cos\theta_1$$

References:

1. Schrader, John Yale, Jr., "An Alternative Approach to Long Range DF Fixing," Naval Postgraduate School Ph.D. Thesis, September 1974.

2. Thompson, K.P. and Kullback, J.H., "Position-Fixing and Position-Predicting Programs for the Hewlett-Packard Model 9830A Programmable Calculator," NRL Memorandum Report 3265, Naval Research Laboratory, Washington, D.C.

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