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INFERENCE OF PROBABILITY OF KILL OF AIR-TO-AIR MISSILES IN VARIOUS ATTACK MODES

THESIS

AFIT/GST/SM/79M-7 Carl J. Vogel
Major USAF

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This research was conducted with the objective of estimating discrete probabilities of successful employment of tactical air-to-air guided missiles in three modes of operation and to estimate confidence intervals about these probabilities of success which did not exceed .20 in length at the 80% level of confidence. A set of proxy data consisting of 34 missile launches was randomly generated to simulate the results of a test series. This set of proxy data was analyzed using Regression Analysis techniques,

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mathematical modeling of the results of the test series as probability density functions, and Bayesian techniques.

Regression analysis techniques did not provide usable results in this application, but the failure may have been due to the nature of the data being analyzed. The events in the flight of a missile were modeled as Beta probability density functions which were statistically combined and inferences were drawn from the distribution representing the overall probability of success. To demonstrate the use of Bayesian techniques to determine a prior distribution from historical data, the basic data was assumed to have been replicated and was then reanalyzed. These analyses did not meet the stated objectives because the means could not be statistically separated at the 80% level of confidence. The lengths of the associated 80% confidence intervals exceeded the objective of less than or equal to .20 in length for all cases when the basic proxy data was analyzed and in one of the three cases when historical data was assumed. Depending on the availability of historical data and the suitability of the approach, a combination of mathematical modeling and Bayesian techniques may meet the stated objectives.

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Preface

This research was performed in an effort to provide improved data reduction techniques for the analysis of data collected in tests of tactical air-to-air guided missiles. Hopefully, the techniques developed will enable the Air Force Test and Evaluation Center (AFTEC) Analysis Branch to meet their objectives of improving the quality of their data reduction efforts.

I wish to express my most sincere thanks to Lieutenant Colonel Charles W. McNichols for his comments and suggestions along the way and for tactfully averting me from near disaster on more than one occasion. I would also like to thank Captain Douglas L. Brazil of AFTEC for suggesting this research topic.

Carl J. Vogel

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Abstract

This research was conducted with the objective of estimating discrete probabilities of successful employment of tactical air-to-air guided missiles in three modes of operation and to estimate confidence intervals about these probabilities of success which did not exceed .20 in length at the 80% level of confidence. A set of proxy data consisting of 34 missile launches was randomly generated to simulate the results of a test series. This set of proxy data was analyzed using Regression Analysis techniques, mathematical modeling of the results of the test series as probability density functions, and Bayesian techniques.

Regression analysis techniques did not provide usable results in this application, but the failure may have been due to the nature of the data being analyzed. The events in the flight of a missile were modeled as Beta probability density functions which were statistically combined and inferences were drawn from the distribution representing the overall probability of success. To demonstrate the use of Bayesian techniques to determine a prior distribution from historical data, the basic data was assumed to have been replicated and was then reanalyzed. These analyses did not meet the stated objectives because the means could not be statistically separated at the 80% level of confidence. The lengths of the associated 80% confidence intervals exceeded the objective of less than or equal to .20 in length for all

cases when the basic proxy data was analyzed and in one of the three cases when historical data was assumed. Depending on the availability of historical data and the suitability of the approach, a combination of mathematical modeling and Bayesian techniques may meet the stated objectives.

INFERENCE OF PROBABILITY OF KILL OF AIR-TO-AIR MISSILES IN VARIOUS ATTACK MODES

I. Introduction

The Air Force Test and Evaluation Center (AFTEC) routinely conducts test programs involving tactical air-to-air guided missiles which are a part of the U. S. Air Force inventory. The purpose of these test programs is to evaluate the effectiveness of modifications to a discrete subsystem of the missile (for example, a fuzing, or a guidance subsystem). A single test program generally does not involve the evaluation of more than one modified subsystem. The tests are designed to evaluate the reliability and operational performance of the missile system. To determine the operational effectiveness of the system, firings are conducted to evaluate the missile system in three modes of operation. These modes include:

1. Target at higher altitude than the receiver (Look-up)

2. Target at a lower altitude than the receiver (Look-down)

3. Target maneuvering in excess of 4 g's (Maneuvering)

It should be noted that while the look-up and look-down modes are mutually exclusive, the maneuvering mode can occur concurrently with either the look-up or look-down modes. This leads to the possible evaluation of a single launch in two different modes of operation.

Because of budgetary constraints, there is a limit to the number of missiles which are fired in any given test program. This figure normally varies from between 20 to 30 missiles fired per test program. In an effort to expand the data base, the missiles are carried on a number of "captive-carry" sorties before they are actually "live-fired". The ratio of captive-carry to live-fire sorties varies by both program and mode of operation from about 6:1 to 13:1. Both reliability and probability of kill data can be extracted from the captive-carry and live-fire sorties.

The probability of kill data obtained from the test sorties is evaluated differently in the two types of sorties. In the live-fire sorties, the missile either does or does not destroy its target, and is rated respectively as a "1" or a "0". In the captive-carry sorties, recordings of various cockpit indications are reviewed in order to subjectively assign a probability that the missile would have destroyed its target had it been launched.

An additional source of disparity in these tests concerns the surrounding conditions of release or simulated release. Because of the limited number of live-fire missions, the missiles are not fired unless all aircraft and missile systems are thought to be in perfect operating order and the weather is ideal. However, the captive-carry launches are simulated under less than ideal conditions.

Once the test program has been completed, the results of all tests are averaged so as to determine a single probability of kill for the modified system. No attempt is made to reduce the test data to reflect a separate probability of kill for the three modes of operation, nor is a confidence interval established. The reason for not establishing separate probabilities of kill or confidence intervals is based on the belief that the relatively small total sample size, compounded by the large number of independent variables involved, could result in erroneous conclusions. And in the case of establishing confidence intervals, the lengths of the intervals would be so large as to be of no value.

Members of the AFTEC Evaluation branch have indicated that identification of a test plan or a refinement in the method of data reduction which would allow the determination of a probability of kill in each of the modes of operation with an appropriate confidence interval would be a valuable contribution to test program reports and the information these reports convey to system users. Further, they feel that an estimation of the kill probability $\pm$ .10 at the 80% confidence level would meet their needs.

Problem Statement

This background leads to the following statement of the problem. The test design and subsequent data reduction techniques used by the AFTEC analysis branch do not permit assigned personnel to either separate accumulated test data to estimate a probability of kill for discrete modes of operation of a subject missile system, nor is it possible to determine a usable confidence level. AFTEC would like to provide the system users with both a probability of kill estimate for each mode of operation and a confidence interval about these probabilities of kill which does not exceed .20 in length at the 80% level of confidence subject to a predetermined number of live-fire sorties and the limitations of the test environment.

Objectives

This study is focused on identifying a solution to the above stated problem.

The primary objectives of the study are:

1. To identify a method of data reduction which will:
   a. allow data to be reduced to give an indication of the expected probability of kill in each of the modes of operation;
   b. allow confidence intervals to be estimated about each of these probabilities which both AFTEC and the system user feel represent a realistic and usable estimate.
2. To reduce the findings to a format which:
   a. uses readily obtainable data;
   b. is easily usable;
   c. is reliable and can be easily interpreted.

Scope/Limitations

The scope of this effort is limited to examining the operational performance evaluations of tactical air-to-air missile systems. It is possible that the findings herein may be applicable to other aspects of the evaluations of these systems, or that they may be transferred to other types of systems. However, these applications will not be addressed.

Utility

The AFTEC air-to-air missile analysis branch is anticipating up to six subsystem evaluations of the Sparrow (AIM-7F) and Sidewinder (AIM-9L) tactical air-to-air missile systems within the next two to three year period. Meeting the previously stated objectives will enable that organization to increase the value obtained in these test programs, and will give the missile system users an improved base of knowledge during the early phases of use of these systems.

Assumptions

It is assumed that there are no constraints to reasonable changes in the amount and type of data collected during an evaluation. That is, if some type of data which is not currently being collected can be demonstrated to be important in the process of data reduction, this reading can be taken in future programs.

In certain portions of this paper, events in a missile's flight are represented by probability distributions. Events specifically addressed include proper launch, launch sequence, guidance, proximity fuzing, impact fuzing, and warhead lethality. It is assumed that random variables representing the probability of success of the above events are statistically independent.

Overview

The remainder of this paper is largely devoted to descriptions of various methods of treating experimental data. The following chapter offers brief descriptions of potentially useful methods which may be applied to the problem at hand. Chapter three addresses the specific applications of these methods in treating the results of missile firings. Results of each of these applications are covered as each method is discussed. Finally, in Chapter four, conclusions are drawn from the results of the research.

II. Potentially Useful Approaches

The thrust of this study is to determine a method of data reduction which will permit the estimation of a probability of kill which is accurate ± .10 at the 80% level of confidence within the constraints of AFTEC policy and the operational limits involved in conducting the tests. Towards this end, the following techniques have been investigated and tested to determine whether they are helpful in reaching the goal.

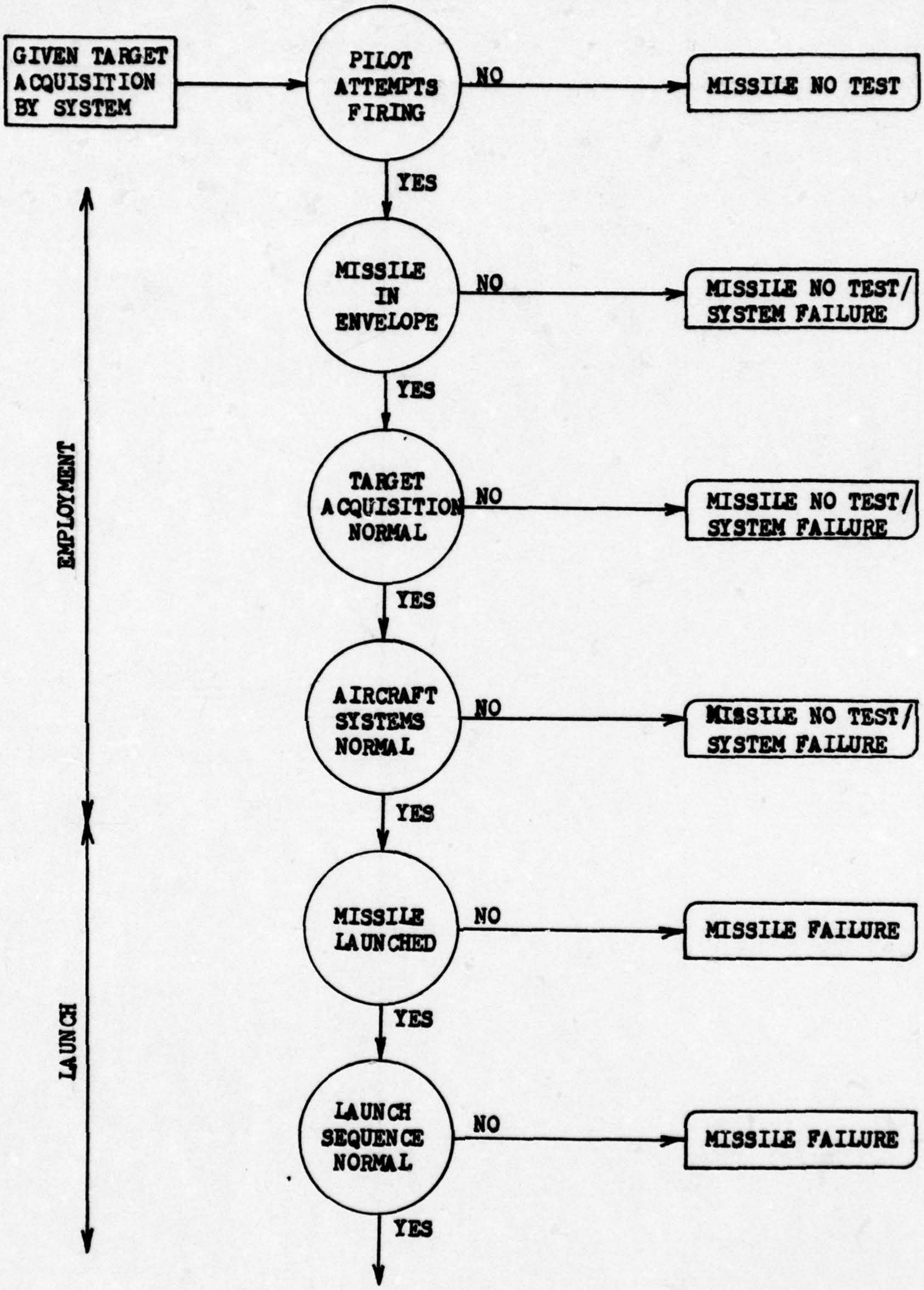
1. Multivariate analysis to include:
   a. linear regression using the raw data
   b. linear regression using dummy variables or classes of data
   c. curvilinear regression
2. Mathematical modeling of the primary phases of the missile's mission
3. Bayesian update using:
   a. history of a subsystem as used in other systems
   b. history of a subsystem which has not been modified for the current test series
   c. subjective input of data

The above listed areas are not mutually exclusive and an acceptable solution may involve a combination of these techniques. For example, a regression technique may prove valuable in predicting the performance of the discrete phases of the mission; the conditional densities of each of these predictors may accurately predict system performance; and Bayesian techniques might be profitably employed to take advantage of prior history of a system or to account for changes made during a test program.

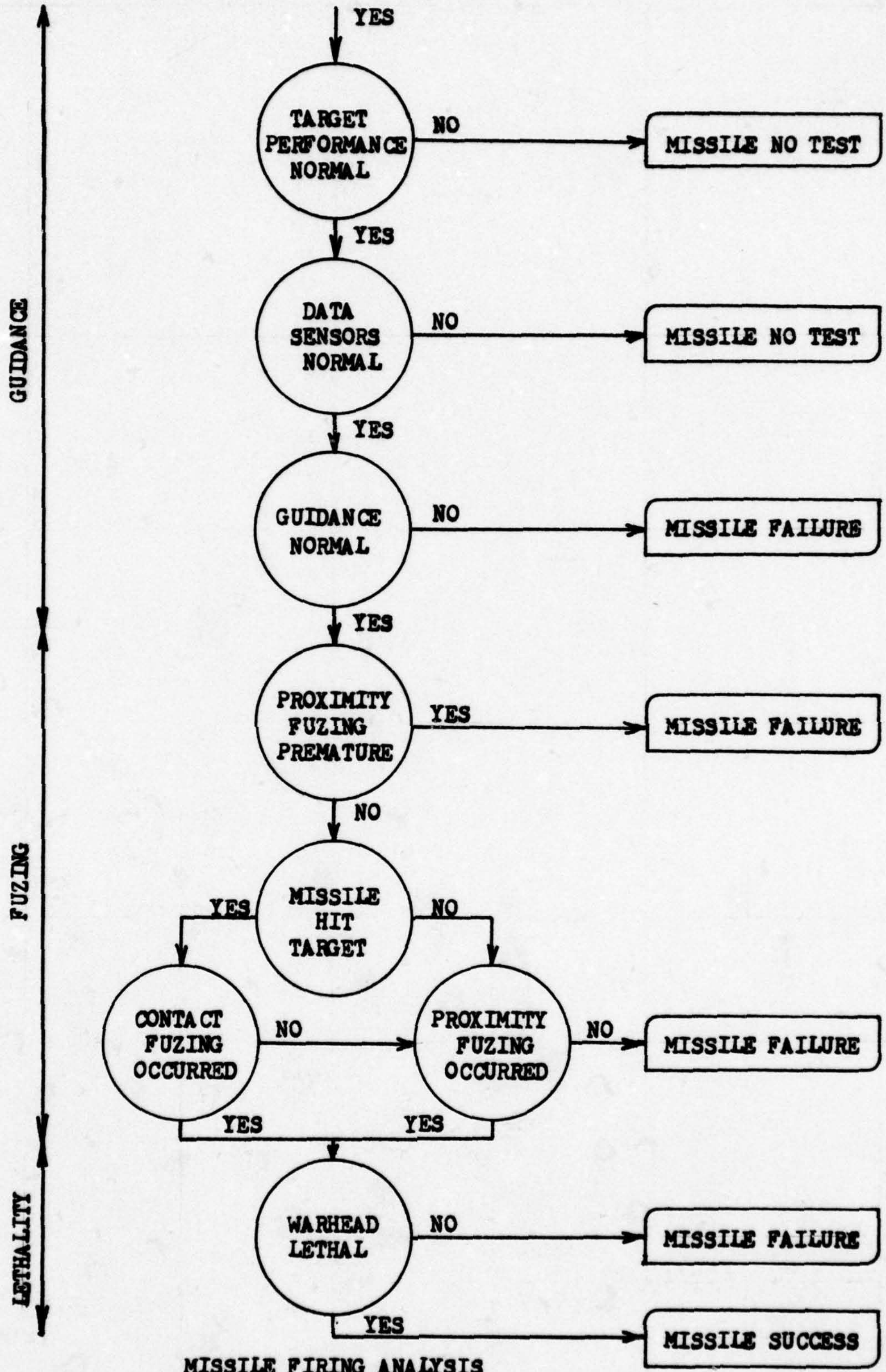
An inherent problem in trying to accommodate these possible approaches is the identification of a framework which is sufficiently flexible. An effort will be made to select specific techniques throughout this study which will readily accommodate other types of treatments.

Because of the large amount of manipulation of data anticipated in this investigation and in the possible combinations thereof, and because of the classification of actual launch conditions and test results, the decision was made to base all preliminary research on a randomly generated set of proxy data. This data consists of 34 flights which have been randomly generated from possible launch conditions as determined from unclassified sources. The data includes 15 launches in the lookdown mode, 19 in the lookup mode, and 19 against maneuvering targets in both modes. A tabulation of this data is included in Appendix 1. Because of the method of generation of this data, it may not bear any resemblance to actual test launch conditions. Similarly, the results of the tests were randomly generated and may not have any correlation with any actual test results. All conclusions have been based on this data, and applications to an actual test program will not be addressed in this document.

For each of the techniques investigated, the overall success or failure of the missile has been examined from two separate points of view. The first, more simplistic approach considers the response variable to be the ultimate success or failure of the system regardless of the cause of failure. Because this approach tends to take a macroscopic view of the missile system operation, it will be referred to as the macro-model. In the second approach the mission of the missile is separated into several nodes. These nodes represent a potential failure point in the mission. Since the evaluation of a missile system may be conducted concurrently with the evaluation of a potential launch platform (for example, the F-15 or F-16), and within a highly controlled test environment, the structure of the model containing the above mentioned nodes has been designed to make allowances for failures in the launch platform or other factors extraneous to the proper operation of the missile which could cause its performance to be unsatisfactory or unmeasurable. These nodes have been included to accommodate possible adaptation of the model to evaluations of the combined launch platform/missile system at a later time, and to recognize the realities of the test environment. However, this study is primarily concerned with the proper operation of the missile itself. The model for this analysis is illustrated in Figure 1. This approach takes a relatively microscopic view of the missile system and will be referred to as the micro-model.



MISSILE FIRING ANALYSIS
FIGURE 1 (Continued next page)



MISSILE FIRING ANALYSIS
Figure 1 (Continued)

As indicated in Chapter one, the nodes of interest will be represented by probability density functions which are assumed to be statistically independent. Some justification of the assumptions of independence is in order. One such specific area requiring explanation is the relationship between guidance and impact. It would seem that the result of the guidance function would directly affect the impact function. However, successful guidance is defined as guidance which places the missile within some predetermined distance of the center of the source which the seeker senses (e.g., the center of the exhaust for an infrared seeker, and the center of the electromagnetic return in the case of a radar seeker). The predetermined distance is generally small relative to the overall physical dimensions of most probable targets. From the above definition of guidance, it can be seen that a direct hit does not imply successful guidance, nor does a miss imply unsatisfactory guidance. Therefore, because the generalized shape of an aircraft is irregular, it is assumed that successful guidance is independent of whether or not the missile impacts the target.

Similarly, the rationale of splitting the functions of the proximity fuze is not readily apparent. The premature portion of the fuze operation is a measure of how well the specific design is able to exclude the influences of clouds and other atmospheric disturbances, as well as background characteristics, and extraneous effects such as "glint" and variable returns peculiar to the sensor, while still sensing its location relative to the target. The node labeled "proximity fuzing occurred" is simply a measure of whether or not the proximity fuze generates a proper pulse to detonate the warhead.

Other areas of the model do not appear to require a justification of the assumption of independence.

The following are brief discussions of each of the techniques employed to predict the performance of an air-to-air missile system.

Regression Techniques

The first approach to be considered involves the use of regression techniques. Ideally, this approach should facilitate the selection of those input conditions which have a significant impact on the probability of success of the system for any phase of the mission or of the entire mission. The results of the analysis lead to the evaluation of coefficients for these significant inputs, and solution of the equation, combining any given set of launch conditions with the coefficients, will yield a prediction of the probability of success for that set of conditions. A confidence interval for this probability can then be determined from the variance-covariance matrix using the following relationship:

$$\bar{Y} \pm t(n-p-1,\ 1-\alpha/2)\ s\sqrt{X_o' C X_o}$$

Where:

C = the variance-covariance matrix

s = square root of the residual mean square

n = the number of observations

p = the number of terms in the regression equation

t = the "t" statistic

X_o = the column vector of the input variables

X_o' = the row vector of the input variables

$\bar{Y}$ = the mean of the response variable

$1-\alpha$ = the desired level of confidence

The specific theory and applications of regression theory are well documented and will not be addressed in this paper. For a treatment of the theory and its development, see Draper and Smith (Ref 3). Another treatment can be found in SPSS: Statistical Package for the Social Sciences, chapters 18, 20, and 21 (Ref 6:276-300, 320-397).

Specific applications of regression techniques considered within this study include the treatment of the data as it was recorded. This approach is not anticipated to be productive because of the large number of independent variables and the wide range of values which they can assume.

A second approach involves the use of dummy variables to reduce the impact of the wide ranges of certain variables. For example, the effect of the launch and target airspeeds may not be as important as the fact that the missile has a speed advantage or disadvantage at launch.

Similarly, the aspect angle can be divided into broad categories which may be more meaningful than the actual value of the angle. Such categories could include launches within a 30° arc about the head-on aspect relative to the target as one dummy value; a 30° arc about the tail-on aspect as another value; and finally, all other aspect angles could be considered as beam attacks receiving a third value.

An attempt has also been made to examine the possible effects of variables not recorded, but which could be crucial to the overall success or failure of the missile system. Such variables might include an estimated time of flight of the missile which may reflect guidance or thrust characteristics. Another such variable is the effective range of the missile or the distance it must actually fly to intercept. This variable is a function of launch range, aspect angle, relative speed, altitude differential, and target maneuver. Finally, the terminal angle of intercept between the missile and the target may directly affect the lethality of the warhead or the firing properties of one or both fuzes. While it is felt that these additional variables may be significant, and, even though an effort has been made to generate them, it is questionable whether such calculations can produce these values with a reasonable degree of accuracy. This is due to the rapid change of all variables during the course of a missile's flight. It should be possible to determine appropriate mathematical statements to accurately reflect these changes, but such an endeavor is beyond the scope of this thesis and the results would be too cumbersome to introduce into a regression program. Efforts to generate additional values are therefore limited to simple approximations which hopefully will reflect the impact of the actual values on the system's operation. The specific explanations are addressed in the chapter on applications.

Finally, methods of curvilinear regression have been employed in an effort to isolate those variables having a higher order contribution to the success of the system.

Mathematical Modeling

Another possible approach in predicting the probability of success for a missile system is to mathematically model the missile's mission in terms of probability density functions. For consistency with other approaches, the mission itself can be modeled as an event, or as the macro-model. The mission can also be separated into a series of discrete events representing the earlier identified nodes as was done in the micro-model. The total number of successes and failures either for the overall mission or for the nodes can then be used as parametric measures which define appropriate probability density functions. In the case of the micro-model, the parameters defining the nodes can be statistically combined to determine the mean and variance of a final distribution. This final distribution can be used as a model to predict the probability of successful operation of the missile system in each mode of operation. In the macro-model, the number of successes and the total number of trials can be used to define a distribution which can also be used to predict the probability of system success.

Problems associated with such an approach are twofold. The first deals with the identification of an appropriate type of probability density function describing a random variable representing the probability of success which is capable of accurately reflecting a wide variety of test results. The second difficulty involves the mathematical determination of the parameters of the resultant density function. For the first of these problems, appropriate distributions may include the Beta or the Normal distributions. In this research no assumptions have been made concerning the normality of the test results, but if such an assumption can be shown to be valid, the approach may prove to be valuable. In the absence of this knowledge, information from a test program can be adequately represented by a Beta distribution. This distribution is defined between zero and one, and has a wide variety of shapes which are determined by the selection of two parameters. The appropriate choice of the input parameters can vary the shape of the distribution to include "U", "J", triangular, and inverted "U" shapes. The uniform density function is also a special case of the Beta family. Another feature associated with the use of the Beta family of distributions is that results of tests having either a "0" or "1" outcome - success or failure - can be used to define the parameters to shape the density function.

The Beta distribution is defined by the equation:

$$f(x) = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & 0 \leq x \leq 1 \\ 0 & \text{Elsewhere} \end{cases}$$

Or alternatively:

$$f(x) = \begin{cases} \frac{\Gamma(N)}{\Gamma(r)\Gamma(N-r)} x^{r-1} (1-x)^{N-r-1} & 0 \leq x \leq 1 \\ 0 & \text{Elsewhere} \end{cases}$$

Where:

$N = \alpha + \beta$ = number of trials

$r = \alpha$ = number of successes

$N-r = \beta$ = number of failures

$\Gamma(Y) = (Y-1)!$ = the Gamma function

The second expression for the Beta function, while less general, has the advantage of including test results directly, and in this application is found to be generally easier to manipulate in some of the derivations which follow. Other properties of the Beta distribution are given below:

$$\text{Mean} = \mu = \frac{r}{N}$$

$$\text{Variance} = \sigma^2 = \frac{r(N-r)}{N^2(N+1)}$$

$$\text{Mode} = M = \frac{r-1}{N-2}$$

Another advantage of the Beta family of distributions is the fact that it is a conjugate family of distributions, and it can be easily manipulated to incorporate additional information by the use of Bayesian techniques. Such manipulations require only the respective addition of the parameters of the sample and prior distributions to determine the posterior distribution.

Since this study is being conducted with attention being paid to the eventual combination of techniques, it should be noted that the primary drawback in using the Beta distribution concerns its use in the Bayesian form where anomalies may occur when no prior knowledge is assumed. Since these are peculiar to the use of the Beta as a conjugate distribution, discussion of these anomalies will be deferred until the subject of Bayesian methods is addressed.

The second problem associated with the use of a probability density function to model the missile system's operation is the determination of the parameters of the distribution representing the probability of successful system operation. For the case of a series model where the individual random variables representing the probability of subsystem operation are statistically independent, the final probability can be expressed as the product of the conditional probabilities, or:

$$P = P_1 \times P_2 \times \dots \times P_n$$

The value of P is determined by the values of random variables P_1, P_2, ... , P_n assume. For the specific case of the model identified earlier in this chapter, the above leads to the following relationship:

$$P_k = P_{la} \times P_{ls} \times P_g \times P_{pp} \times P_f \times P_{le}$$

Where:

Pn = a random variable representing the probability of success at the nodes with the subscripts:

k = kill

la = launch

ls = launch sequence

g = guidance

pp = no premature proximity fuzing

f = fuzing

le = warhead lethality

This statement, however, ignores the presence of two fuzes and their possible parallel nature in the impact loop. Since it is common design practice to equip an air-to-air missile with both proximity and impact fuzes, the measure of success of the fuze subsystem is a function of the proper operation of both fuzes, modified by the probability of the missile impacting the target. In the case of no impact, the probability is simply the probability of the proper operation of the proximity fuze multiplied by the complement of the probability of impact in series with the remainder of the model. In the case where the missile impacts the target, the probability of proper fuzing is the complement of the probability of the failure of both fuzes multiplied by the probability of impact. The successful fuzing for the missile is then a function of whether or not the missile impacts the target, and the proper operation of both the proximity and contact fuzes, or:

$$P_f = (1-P_i)\ (P_{fp}) + (P_i)\ (1-(1-P_{fp})\ (1-P_{fc}))$$

$$= P_i P_{fc} - P_i P_{fc} P_{fp} + P_{fp}$$

Where:

P_i = a random variable representing whether or not the missile impacts the target

P_{fp} = a random variable representing proper operation of the proximity fuze

P_{fc} = a random variable representing the proper operation of the contact fuze

The overall model for success of the entire missile system then becomes:

$$P_k = P_l\ P_{ls}\ P_g\ P_{pp}\ [P_i P_{fc} - P_i P_{fc} P_{fp} + P_{fp}]\ P_{le}$$

This function can be relatively easily approximated using Monte Carlo techniques. This approach involves the generation of a Beta distributed random variate from each of the distributions representing the nodes and combining these variates as indicated in the above model. The resultant probability represents one variate from the final density function.

Iteration of this process will yield a number of variates from the final density function. The mean and variance of this sample can be calculated and inferences can be drawn as to the nature of the density function for P_k. The level of confidence in the information thus obtained is a function of the number of iterations and the underlying assumptions.

With only the assumption that the distribution has a finite mean and variance, the use of Chebyshev's Theorem is appropriate. Chebyshev's Theorem is also valuable in defining the worst case situation in the event that no further limiting assumptions are warranted. It might be possible to use the Central Limit Theorem to justify an assumption of normality if the underlying distributions are all assumed to be "well-behaved". With minimal assumptions about the nature of the underlying distributions, a density function representing a sum of independent random variables will converge asymptotically to a normal distribution, and a product of independent random variables will converge to a log normal distribution.

The relative advantages of using Chebyshev's Theorem over the Central Limit Theorem lies in its applicability regardless of the nature of the final distribution. Thus, it does not open the door to criticisms based on assumptions made to reduce the size of the sample required. However, relaxing the initial assumptions carries the cost of significantly larger number of data points to arrive at similar levels of confidence if using the assumption of normality. For example, a 90% confidence level that the mean of the simulation is within a given interval about the true mean using Chebyshev's Theorem requires 3.72 times the number of samples which would be required using the Central Limit Theorem as justification for the assumption of normality. A 80% confidence interval would require 3.05 times the number required for the same interval and level of confidence using the assumption of normality.

The analytic solution of the model is somewhat more cumbersome than the Monte Carlo approximation, and it is not as easily changed, should changes in the overall model be desired. However, the method has the advantage of not inherently introducing uncertainty as the Monte Carlo approximation does. The method used to arrive at a mean and variance of the function representing system success involves the use of expected values of the model determined above. In the earlier defined model, the random variable representing the success of the system (P_k) is represented as a product of probabilities of success of the individual nodes except for that portion of the model which represents fuzing operation. The entire impact/fuze loop of the model can be considered to be in series with the remainder of the model. The following derivations will find the desired parameters of the loop so it can be treated as a series component of the model. The following basic relationships will be used to determine the expected value (the mean) and the variance of the earlier proposed mathematical model:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \mu_x$$

$$E(X_1 X_2) = E(X_1)\ E\ (X_2)$$

(For independent X_1, X_2)

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

$$E(CX) = C\ E(X)$$

(C = a constant)

$$V(X) = E(X^2) - (E(X))^2$$

Where:

$E(X)$ = the expected value of the random variable, X.

$V(X)$ = the variance of the random variable, X.

As demonstrated earlier, the impact/fuzing loop can be represented as follows:

$$P_f = P_i\ P_{fc} - P_i\ P_{fc}\ P_{fp} + P_{fp}$$

The expected value of this random variable is:

$$E(P_f) = E(P_i\ P_{fc} - P_i\ P_{fc}\ P_{fp} + P_{fp})$$

$$= E(P_i\ P_{fc}) - E(P_i\ P_{fc}\ P_{fp}) + E(P_{fp})$$

Since all events have been assumed to be statistically independent:

$$E(P_f) = E(P_i)\ E\ (P_{fc}) - E\ (P_i)\ E\ (P_{fc})\ E\ (P_{fp}) + E\ (P_{fp})$$

$$= \mu_i\ \mu_{fc} - \mu_i\ \mu_{fc}\ \mu_{fp} + \mu_{fp}$$

The overall model is:

$$E(P_k) = E(P_l\ P_{ls}\ P_g\ P_{pp}\ P_f\ P_{le})$$

Again, because of independence:

$$E(P_k) = E(P_l)E(P_{ls})E(P_g)E(P_{pp})E(P_f)E(P_{le})$$

$$= \mu_l\ \mu_{ls}\ \mu_g\ \mu_{pp}\ \mu_f\ \mu_{le}$$

Substituting the expression for the expected value of the impact/fuzing loop:

$$E(P_k) = \mu_{la}\ \mu_{ls}\ \mu_g\ \mu_{pp}(\mu_i\mu_{fc} - \mu_i\ \mu_{fc}\mu_{fp} - \mu_{fp})\ \mu_{le}$$

The derivation of the variance is more complicated than that for the mean, and it will be approached in two separate parts. First, a general derivation of the variance of the product of independent Beta distributions will be demonstrated. Second, the variance of the impact/fuzing loop will be derived.

The variance of the product of n random variables is given by the following expression:

$$V(\prod_{i=1}^{n} P_i) = E((\prod_{i=1}^{n} P_i)^2) - (E(\prod_{i=1}^{n} P_i))^2$$

Because of independence, and the related fact that the covariance between the independent variables equals zero, this expression can be reduced to:

$$V\left(\prod_{i=1}^{n} P_i\right) = \prod_{i=1}^{n} E(P_i^2) - \left[\prod_{i=1}^{n} E(P_i)\right]^2$$

Using the definition of variance to evaluate the $E(P^2)$ terms:

$$V(X_i) = E(X_i^2) - (E(X_i))^2$$

Therefore:

$$E(X_i^2) = V(X_i) + (E(X_i))^2 = \sigma_i^2 + \mu_i^2$$

Finally:

$$V\left(\prod_{i=1}^{n} P_i\right) = \prod_{i=1}^{n} (\sigma_i^2 + \mu_i^2) - \left[\prod_{i=1}^{n} \mu_i\right]^2$$

The mean and variance of the Beta Distribution have been given earlier as:

$$\sigma^2 = \frac{r(N-r)}{N^2(N+1)} \qquad \mu = \frac{r}{N}$$

Substituting these expressions into the equation for the variance of a product of independent Beta random variables yields:

$$V\left(\prod_{i=1}^{n} P_i\right) = \prod_{i=1}^{n} \left[\frac{r_i(N_i-r_i)}{N_i^2(N_i+1)} + \left(\frac{r_i}{N_i}\right)^2\right] - \left[\prod_{i=1}^{n} \mu_i\right]^2$$

$$=\prod_{i=1}^{n} \left[\mu_i \frac{N_i - r_i + r_i N_i + r_i}{N_i(N_i+1)} \right] - \left[\prod_{i=1}^{n} \mu_i\right]^2$$

$$=\prod_{i=1}^{n} \left[\mu_i \frac{(r_i+1)}{(N_i+1)} \right] - \left[\prod_{i=1}^{n} \mu_i \right]^2 .$$

The variance of the impact/fuzing loop will be derived by defining three additional variables as follows:

$$Y_1 = P_i P_{fc}$$
$$Y_2 = P_i P_{fc} P_{fp}$$
$$Y_3 = P_{fp}$$

The variance of the loop is:

$$V(Y_1 - Y_2 + Y_3) = E((Y_1 - Y_2 + Y_3)^2) - (E(Y_1 - Y_2 + Y_3))^2$$

$$= E(Y_1^2) + E(Y_2^2) + E(Y_3^2) - 2E(Y_1 Y_2) + 2E(Y_1 Y_3) - 2E(Y_2 Y_3)$$

$$-[(E(Y_1))^2 + (E(Y_2))^2 + (E(Y_3))^2 - 2E(Y_1)E(Y_2)$$

$$+2E(Y_1)E(Y_3) - 2E(Y_2)E(Y_3)]$$

$$= E(Y_1^2) - (E(Y_1))^2 + E(Y_2^2) - (E(Y_2))^2 + E(Y_3^2) - (E(Y_3))^2$$

$$-2E(Y_1 Y_2) + 2E(Y_1)E(Y_2) + 2E(Y_1 Y_3) - 2E(Y_1)E(Y_3)$$

$$-2E(Y_2 Y_3) + 2E(Y_2)E(Y_3)$$

The first three pairs of terms on the right-hand side of the final equation can be seen to be the variances of the three previously defined variables. Each can be evaluated by using the general equation for the variance of a product which was derived earlier. The last three pairs of terms represent the three covariance terms for the three possible pairs of variables which will be evaluated by substituting their initial definition into each expression.

$$E(Y_1Y_2)=E(P_i^2P_{fc}^2P_{fp})=E(P_i^2)E(P_{fc}^2)E(P_{fp})$$

Since, from the definition of variance:

$$E(X^2) = V(X) + (E(X))^2$$

Therefore, $E(Y_1Y_2)$ can be evaluated as:

$$E(Y_1Y_2) = (\sigma_i^2+\mu_i^2)(\sigma_{fc}^2+\mu_{fc}^2)\mu_{fp}$$

Also, by definition:

$$E(Y_1)E(Y_2)=(\mu_i\mu_{fc})(\mu_i\mu_{fc}\mu_{fp})=\mu_i^2\mu_{fc}^2\mu_{fp}$$

Similarly:

$$E(Y_1Y_3) = \mu_i\mu_{fc}\mu_{fp}$$

$$E(Y_1)E(Y_3) = \mu_i\mu_{fc}\mu_{fp}$$

$$E(Y_2Y_3) = \mu_i\mu_{fc}(\sigma_{fp}^2+\mu_{fp}^2)$$

$$E(Y_2)E(Y_3) = \mu_i\mu_{fc}\mu_{fp}^2$$

Noting that $E(Y_1Y_3) = E(Y_1)E(Y_3)$ and substituting the remaining values into the equation for the variance of the impact/fuze loop yields:

$$V(P_f)=\sigma^2_{i,fc}+\sigma^2_{i,fc,fp}+\sigma^2_{fp}-2[(\sigma_i{}^2+\mu_i{}^2)(\sigma_{fc}{}^2+\mu_{fc}{}^2)(\mu_{fp})]$$

$$+2\mu_i{}^2\mu_{fc}{}^2\mu_{fp}-2[(\mu_i\mu_{fc}(\sigma_{fp}{}^2+\mu_{fp})]+2\mu_i\mu_{fc}\mu_{fp}{}^2$$

$$+2\mu_i\mu_{fc}\mu_{fp}-2\mu_i\mu_{fc}\mu_{fp}$$

$$=\sigma^2_{i,fc}+\sigma^2_{i,fc,fp}+\sigma^2_{fp}-2\mu_{fp}[\sigma_i{}^2\sigma_{fc}{}^2+\sigma_i{}^2\mu_{fc}{}^2+\mu_i{}^2\sigma_{fc}{}^2]$$

$$-2[\mu_i\mu_{fc}\sigma^2_{fp}]$$

The above equations for the mean and the variance of the impact/fuze loop can be solved to obtain an N and r term for the loop which can be incorporated into the calculations of the mean and variance for the overall model.

Bayesian Inference

One final method which has been examined to determine whether it has value in drawing inferences from missile tests involves the use of Bayesian techniques permitting the inclusion either of prior knowledge of a sybsystem's performance or of subjective estimates of that performance. As was noted earlier, the Beta family of distributions was selected to represent test results partially because of the fact that it is a "conjugate prior distribution". According to Hayes and Winkler, the desirable characteristics of such distributions include:

1. Mathematical tractability - the type of distribution

should be one for which it is relatively easy to specify a posterior distribution given a prior distribution and a liklihood function; and that the posterior is a member of the same family as the prior distribution;

2. Richness - the family of distributions should be able to represent a wide variety of states of information in terms of central tendency and dispersion, as well as a variety of shapes;
3. Ease of interpretation - the family should be readily interpretable to the person whose prior information is of interest as well as to the analyst (Ref 4:459).

As was previously mentioned, the Beta family of distributions can be used to represent a wide variety of states of information, and the shapes available are adequate to express these states in most common usages. It is also as easily interpretable as most other distributions and the influence of varying the shape parameters is easily envisioned by use of equations relating these parameters to the moments of the distribution. Finally, the family is mathematically tractable in that an updated Beta distribution will yield a Beta distribution, and the process is simple because addition of prior and sample parameters respectively define the posterior distributions. For a detailed development of the points relating to mathematical tractability, see Tummala (Ref 7), pp. 412, 414, and Miller and Freund (Ref 5), chapters 8 and 9.

Deferred until this time were possible anomalies occurring when the use of the Beta distribution is unaccompanied by previous test data and the decision is made not to use subjective prior information. The objective in this case is to convey an informationless or "diffuse" prior state. If the true state of knowledge about the system reflected an equal probability that the probability of success could assume any value between zero and one, then a uniform distribution (a member of the Beta family) would seem to reflect the most accurate prior information. However, because of the nature of the problem at hand, the assumption of a uniform prior distribution can bias the test results. Two factors contribute to this bias. First is that in the Bayesian process, the posterior mean always lies between the prior mean and the sample mean. For purposes of illustration, let us assume for the moment that a missile system is acceptable only if it is successful at least 50 percent of the times it is fired, and make the simplifying assumption that the system is adequately represented by six nodes in series. The success at each node, then, must be greater than or equal to .5 and in actuality, the success at each node must be in the vicinity of the sixth root of .5, or around .89. If the probabilities of success at each node are successively shifted towards the mean of the uniform prior distribution, the overall probability of success must be lower than the treatment of the raw test data would have reflected. If a mean of the probability of success for a missile system in the region of .5 to .7 is a reasonable goal, the assumption of a uniform prior distribution will lead to a conservative estimate of the mean and may lead, in turn, to the rejection of an acceptable system.

The second source of bias is that the shift of the most probable region of the posterior distribution to a lower range is aggravated by a small number of data points. This is because the parameters of the uniform prior distribution are always N=2, r=1; and the simple addition involved in the updating process tends to reflect the influence of the prior parameters on the posterior distribution to a greater degree as the size of the sample decreases and/or as the number of successes approaches either zero or the number of trials.

An alternative approach is available to reflect the absence of prior information. This procedure is to allow the prior parameters N and r to approach zero. In practice it is practical to let N = r = 0 to avoid the perturbations caused by small values of the prior parameters whose influence we are trying to damp. With zero weighting of the prior information, the posterior distribution directly reflects the sample data. This practice, however, assumes a prior distribution which is undefined because as the parameters approach zero the distribution is "U" shaped and at x = 0 and at x = 1 the function becomes undefined.

In defense of this practice is the pragmatic acceptance that is accurately reflects the information available.

Stated more eloquently, "But it seems to us that the real test of a diffuse prior distribution in Bayesian inference and decision is whether or not it affects the posterior distribution. This is because ... the ultimate aim of the Bayesian is to use the posterior distribution in an inferential or decision making process." (Ref 4:467).

For the above listed reasons, and following the lead of the authors of the above quote, the absence of prior information will be represented as a Beta distribution with both parameters equal to zero throughout this paper.

III. Applications

In this chapter specific applications of the approaches which were developed in Chapter two will be addressed. The results in each area will be included as the approach is developed, and the last section of the chapter will discuss the issue of sensitivity analysis.

Regression Analysis

The proxy data which is included in Appendix 1 was analyzed using several of the regression techniques outlined in Chapter two. The objective of this effort was to identify which of those recorded values are important in predicting the success of a missile system for a particular mode of operation. The criterion selected to measure the effectiveness of each trial was the value of the Multiple Correlation Coefficient (R^2). This value would give an indication of the variance of the response variable which had been accounted for by the regression, and, hence, an indication if desired confidence interval length could be achieved. For initial research, the success or failure of the guidance, warhead lethality, and the overall flight were considered as response variables. All nodes were not considered individually in the interests of reducing computer analysis requirements.

The actual data reduction was accomplished using the Statistical Package for the Social Sciences (SPSS) Regression routine with the stepwise method of variable inclusion. As anticipated, the regression using the proxy data in raw form was not particularly effective. The values of R^2 obtained by using SPSS default levels for the inclusion of independent variables was below .4 for all cases. Using rather low levels for the inclusion of variables ("F" ratio equal to .5, and the tolerance, defined in the SPSS manual as the variance of an independent variable being considered for inclusion to the variance not explained by the variables already in the equation, equal to .05) decreased the value of R^2 by about 20 percent for the various modes of operation, while generally deleting three to five variables from the regression equation. After the initial approach, efforts included the computation of dummy variables for the missile airspeed advantage/disadvantage at launch, blocking differential altitudes into 2000 foot increments, grouping launch ranges into five nautical mile increments, and grouping the aspect angles into head-on (180°±30°), tail (0°±30°), and beam (all remaining). This approach was not found to produce significantly more meaningful results than use of the raw data. The logarithms of all variables were computed and regressed with the raw data and the dummy variables with no significant change in the results. The final regression technique which was employed involved a simplified attempt to generate variables representing the range the missile actually covers in its flight, the time of flight, and the terminal offset angle. Initially, several simplifying assumptions were made. The first was that the line of flight and airspeed of the target aircraft did not change during the time the missile was in flight. Secondly, the missile was assumed to depart the launch aircraft on its terminal line of flight.

Different assumptions were made concerning the airspeed including constant airspeed, and constant acceleration. With the above assumptions a triangle can be solved to determine the effective range of the missile, its time of flight, and terminal offset angle. These variables were introduced into the regression routine along with those variables which appeared to be significant based on the results of previous trials. Attempts were made to include a decreasing acceleration for the missile and a constant rate turn for the target. During the derivation of the necessary relationships, the results for the constant acceleration model were analyzed, which led to abandoning regression techniques. Generally, the confidence intervals obtained were too large to be of value in a practical application. Although the regression analysis was based on proxy data, it was felt that any specific techniques which were developed could not be adequately tested because of the limited availability of both the histories of actual test programs and of operational launches. For these reasons it was felt that further efforts expended in this area would have marginal returns and probably could not be validated and the approach was dropped.

Mathematical Modeling

To facilitate the analysis of the proxy data using density functions to represent the operation of a missile system, the data contained in Appendix 1 can be reduced as shown in Table 1. Each column of entries represents the results of a different mode of operation, with the exception of the fourth column, which represents the overall test program results. Each pair of entries represents the number of successes for a given phase of the missile's flight, as well as the number of missiles surviving to that point of the flight. Each of these pairs of entries was used to define a Beta probability density function for the probability of successful operation at each node in the micro-model as defined in Chapter 2. The final entry for each mode of operation is the number of successes for the warhead lethality node and the number of attempted launches. These final entries were used as parameters for the macro-model approach of modeling the entire test series with a single density function.

Efforts to analyze the proxy data have involved the use of both Monte Carlo simulation and the analytic approach to the model which was derived earlier. Both approaches involved the analysis of the micro- and the macro-model. Chronologically, the Monte Carlo analysis was well underway before it was apparent that an analytic solution could be reached. As discussed in Chapter 2, the use of a simulation approach will inherently introduce some degree of uncertainty into the final result. The Monte Carlo approximation was

Table I
Success of Missile System at Nodes of Operation

	Modes of Operation			
Node	Lookdown	Lookup	Maneuver	Overall
Launch	13/15	19/19	18/19	32/34
Launch Seq.	12/13	18/19	16/18	30/32
Guidance	10/12	17/18	14/16	27/30
No Premature Prox. Fuze	9/10	16/17	12/14	25/27
Impact	4/9	4/16	4/12	8/25
Contact Fuze	4/4	3/4	3/4	7/8
Proximity Fuze	8/9	15/16	11/12	23/25
WH Lethality	8/9	14/16	12/12	22/25
Overall	8/15	14/19	12/19	22/34

valuable in validating the analytic model, but because of the inherent uncertainty of the Monte Carlo approach, the analytic approach is favored for an actual application.

The specific methods used in the Monte Carlo approach involved the generation of 1000 random variables according to the model developed for the micro-modeling approach. Another 1000 random variables were generated from a Beta density function using the overall success/failure results of each mode of operation as parameters of the distribution. The variance and mean of each of these populations were determined using a computer library function. Confidence intervals were determined by selecting the 101st and 900th largest values from the populations as the limits of the confidence intervals. Chi-square goodness-of-fit tests were then conducted on each set of random variates to see if they seemed to be from a Beta distribution, a normal distribution, or a log-normal distribution. The procedure used a computer library function which evaluated each of the candidate distributions for ten equiprobable cell frequencies.

In the analytic approach, the mean and variance were determined by using the relationships developed in Chapter 2. Confidence intervals were estimated by using a Simpson's rule approximation of the appropriate function. The limits were determined as being those values which excluded .10 of the area from each side of the distribution. Excluding equal areas from both ends of the area beneath the function yields the shortest confidence interval only for symmetric distributions. In this case, the normal distribution is the only distribution which is always symmetric. The Beta distribution is symmetric only when the mean is equal to .5, and the log-normal distribution is never symmetric. Therefore, it will not generally be true that the confidence intervals obtained when using the Beta or Log-normal distributions are the shortest which exist. However, given the nature of the problem at hand, with means generally located in the .5 to .7 range, it is not expected that the skew of the resultant distributions will significantly affect the length of the confidence intervals. This effect will be more accentuated as the mean is moved further from .5 and as the number of samples decreases. Tests of the Beta distribution with a mean of .75 and ten trials indicated that the length of the shortest 80 percent confidence interval obtainable did not vary by more than .01 from the confidence interval obtained by excluding equal areas from both ends of the distribution.

The actual reduction of the data was accomplished on a TI Programmable-58 calculator. A listing of the program and instructions for its use are included in Appendix 2.

A listing of the results of both the Monte Carlo and the analytic analyses are included in Table II. Several points about these results warrant comment. First, some inferences can be drawn as to the nature of the density function for P_k. Second, the means are not equal for the two types of model for any mode of operation. Thirdly, the proximity of the means and the overlap of the confidence intervals do not permit statistical separation of the means. Finally, in no case does the resultant 80% confidence interval about the mean meet the objective of being less than or equal to .20 in length. Although the data are hypothetical, the results of actual tests would be about the same order of magnitude, and confidence intervals of about the same length would result. The failure to meet the desired length of confidence interval as well as the issue of sensitivity analysis will be addressed later in this chapter.

Table II
Results of Analysis of Proxy Data

Micro-Model			Lookdown	Lookup	Maneuver	Overall
			Mode of Operation			
Analytic	Variance		.01477	.01003	.01166	.00645
Analytic	Mean		.500	.699	.592	.610
Analytic	80% CI	Beta	.34-.66	.56-.83	.45-.73	.50-.72
Analytic	80% CI	Normal	.34-.66	.56-.83	.45-.73	.51-.72
Analytic	80% CI	Log Normal	.35-.66	.57-.83	.46-.73	.51-.72
Simulated	Variance		.01489	.00942	.01196	.00642
Simulated	Mean		.507	.696	.586	.606
Simulated	80% CI		.35-.67	.57-.82	.44-.73	.50-.71
Simulated	X^2	Beta	2.80	5.80	9.70	7.30
Simulated	X^2	Normal	3.78	9.20	16.26	7.02
Simulated	X^2	Log Normal	51.80	32.14	54.68	26.16
Macro-Model						
Analytic	Variance		.01556	.00969	.01163	.00652
Analytic	Mean		.53	.74	.63	.65
Analytic	80% CI	Beta	.37-.70	.60-.86	.49-.77	.54-.75
Analytic	80% CI	Normal	.37-.70	.61-.87	.50-.76	.54-.75
Analytic	80% CI	Log Normal	.39-.70	.61-.86	.50-.77	.55-.75
Simulated	Variance		.01596	.00980	.01181	.00623
Simulated	Mean		.529	.732	.632	.643
Simulated	80% CI		.36-.69	.60-.85	.49-.77	.54-.74
Simulated	X^2	Beta	4.86	12.66	10.78	9.44
Simulated	X^2	Normal	5.62	33.5	9.5	4.28
Simulated	X^2	Log Normal	50.62	83.0	35.82	9.3

Based on the results of the Chi-square tests, the null hypothesis that the randomly generated values are Log-normally distributed can be rejected at the 99 percent level of confidence in seven of the eight cases. On the other hand, the hypothesis that the populations are Beta distributed cannot be rejected in any of the eight cases at the same level of confidence. The hypothesis that the populations are normally distributed can be rejected in only one case. Because of these results and because the tests against a Beta distribution yielded lower values of the Chi-square statistic in five of the eight cases, the Beta distribution is favored to represent the distribution describing P_k for the hypothesized test results.

The difference between the means is accounted for by the difference in the treatment of the impact/fuzing loop in the two methods of computation. In the case of the macro-model, the loop is simply reduced to the number of missiles surviving until the start of the loop and those surviving the flight through the loop. These values are important only in the sense that survival through all subsequent phases can be no greater than survival through any given phase. That is, these values are only indirectly represented in the final distribution function. The micro-model of the loop takes the effects of impact and redundant fuzes into account, so that the expected number of missiles surviving the loop is not a linear function of impact or the operation of either or both fuzes. The conditional nature of the activities represented in the loop dictates that the mean of the loop be less than or equal to the greater value of the mean of proximity fuzing and the product of the means of the probability of impact and the probability of contact fuzing. The relationship of these functions to the mean of the loop is shown in Figure 2, with the mean of the loop being represented as a function of the proximity fuze operation and impact/contact fuze operation.

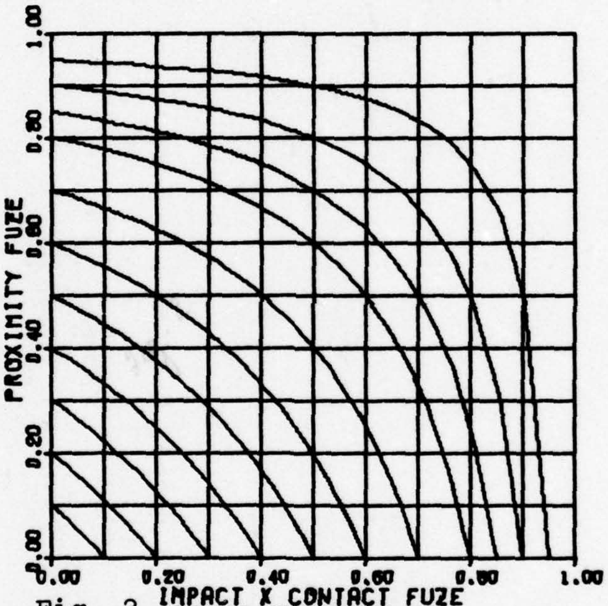


Fig. 2 MEAN OF IMPACT/FUZING LOOP AS A FUNCTION OF IMPACT AND FUZE OPERATION

With the differences in handling the available information, it is probable that some shift in the mean will occur and the case where the means are equal is the exception rather than the rule. Because of the more detailed nature of the micro-model, the value of inferences based upon the single distribution of the macro-model is questionable.

The third point warranting comment is the fact that the means of the resultant distributions, regardless of mode of operation or method of computation, are not statistically unequal. This is illustrated in Figure 3 which shows the relative spans of the intervals. This may be a result of the data being analyzed. However, the data is felt to be reasonably representative of the results of a test series. For this reason, the statistical hypothesis that the means are unequal cannot be rejected, and a further attempt to gather additional evidence to lead to its rejection was made.

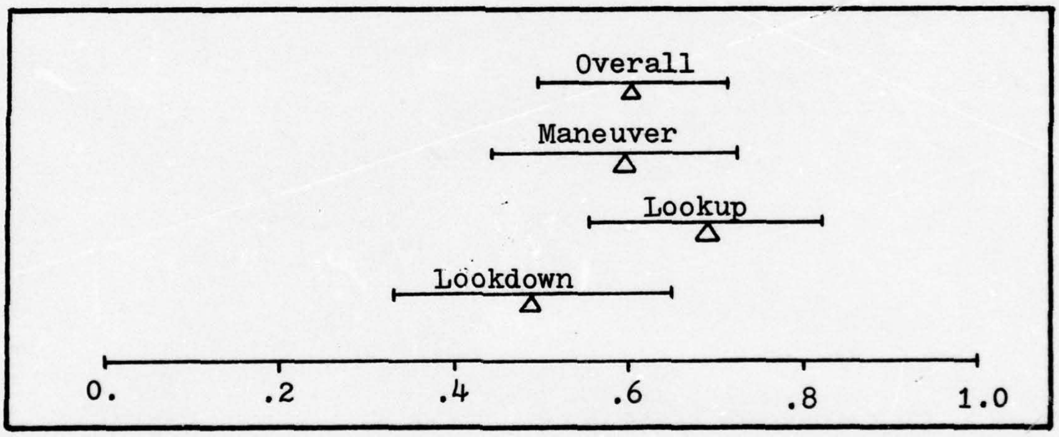


Fig. 3 Means and 80% Confidence Intervals.

The fourth comment relating to the analysis of the proxy data is that this method did not meet earlier stated objectives. This serves well as a transition into Bayesian methods which, depending on the suitability of their application, may enable us to meet the objectives.

Bayesian Techniques

There are two broad categories of information which may be used as prior information in the models being used. These include the use of historical test information and subjective information. While both categories of information are manipulated in the same manner to physically determine a prior distribution, there is considerable difference in the methods used to obtain each type of information, and as will be shown, each can have different effects on the posterior distributions. For this reason, the collection of each type will be addressed separately.

At this point, a note is probably in order concerning the differences between historical and subjective data sources. The two terms tend to define absolutes in a spectrum of degrees of objectivity versus subjectivity, while in practice it is improbable that either extreme is ever encountered. For example, a subjective decision determines what data is to be used from historical sources, and subjective inputs may be based on historical knowledge of very similar subsystems, the results of captive carry sorties, or knowledge of laboratory demonstrations of modified subsystems under evaluation. Care must be taken in the selection of data. This is particularly true in the case of using historical data to determine a prior distribution for the macro-model. This study has been primarily addressed to the incremental evolution of a missile, or to those evaluations which are concerned with the operation of a system already in use, in which some subsystem has been modified. With this assumption as background, it is questionable whether historical data can be applied to the case of the missile system being modeled by a single distribution, since it is reasonable to expect some modification of the subsystems. On the other hand, the inclusion of historical data in the case of the micro-model seems to be a valuable approach since it is improbable that all of the subsystems will be modified.

The use of historical data in determining a prior distribution is a relatively simple process, and can consist of little more than the collection of data from previous test efforts or from a history of operational launches. As noted in Chapter two, the process of defining the prior distribution for the Beta family simply consists of using values for the number of successes (r) and the number of trials (N) in expressions for the Beta function's parameters. The update process requires the respective addition of the number of successes and the number of trials included in the prior and sample information. These sums define the shape parameters for the posterior distribution, and inferences can be drawn from the posterior distribution. In the case of the micro-model, the posterior for any or all nodes can be used as inputs for the formulae developed in Chapter two.

To explore the usefulness of Bayesian methods, the proxy data set described earlier has been doubled in size by replicating each record and re-analyzed. The results are presented in Table 3. However, some of the statistics have been deleted for the sake of clarity. Briefly, the results of the simulation were that the hypothesis that the population of random variables were Log-normally distributed was rejected in seven of the eight cases at the 99 percent level of confidence. The Chi-square statistic for the Beta distribution was lower than that for the normal in seven of the eight cases. These results still tend to favor the Beta distribution as more accurately reflecting the distribution for P_k.

Table III
Results of Analysis of Proxy Data After Simulated Replication

		Mode of Operation			
	Micro-Model	Lookdown	Lookup	Maneuver	Overall
Analytic	Variance	.00760	.00514	.00597	.00327
	Mean	.50	.70	.59	.61
	Beta 80% CI	.38-.61	.60-.79	.49-.69	.53-.68
	Length of CI	.23	.19	.20	.15
Simulated	Variance	.00730	.00547	.00555	.00336
	Mean	.50	.70	.59	.61
	80% CI	.39-.61	.60-.79	.49-.68	.53-.68
	Beta X^2	8.6	6.92	6.58	8.6
	Macro-Model				
Analytic	Variance	.00803	.00497	.00597	.00331
	Mean	.53	.74	.63	.65
	Beta 80% CI	.42-.65	.64-.83	.53-.73	.57-.72
	Length of CI	.23	.19	.20	.15
Simulated	Variance	.00822	.00520	.00581	.00320
	Mean	.53	.73	.63	.64
	80% CI	.41-.65	.64-.82	.53-.73	.57-.72
	Beta X^2	15.24	13.48	6.98	6.84

This example would represent a best case situation of two consecutive evaluations of a system unmodified between evaluations. In practice, it is probable that at least one subsystem would be modified between evaluations, and there would be no historical data to represent that subsystem. Hence, the results would reflect a somewhat larger confidence interval than those shown. While there is still overlap between each of the modes of operation, the confidence intervals of all modes appear to be approaching the objective of .20 in length. The overlap between the look-up and look-down is down to a very small value. The confidence intervals and means are shown in Figure 4.

At this point in the description of applications, a short diversion will be made to illustrate the effects of using a uniform prior distribution rather than the "diffuse" prior as discussed in Chapter two. For purposes of illustration, the analytic approach will be used and the basic data of 34 total launches will be analyzed. The means and confidence intervals from Figure 3 have been reproduced at the top of Figure 5 for purposes of comparison with the same data treated with a uniform prior distribution. It can be seen that the assumption of a uniform prior distribution for the case of the individual modes of operation shifts the means to a value .16 - .19 lower than does the treatment of the same data with a prior distribution with $N = r = 0$ (the "diffuse" prior). Because of the much larger sample size, the mean for all launches dropped only .10. The effects of

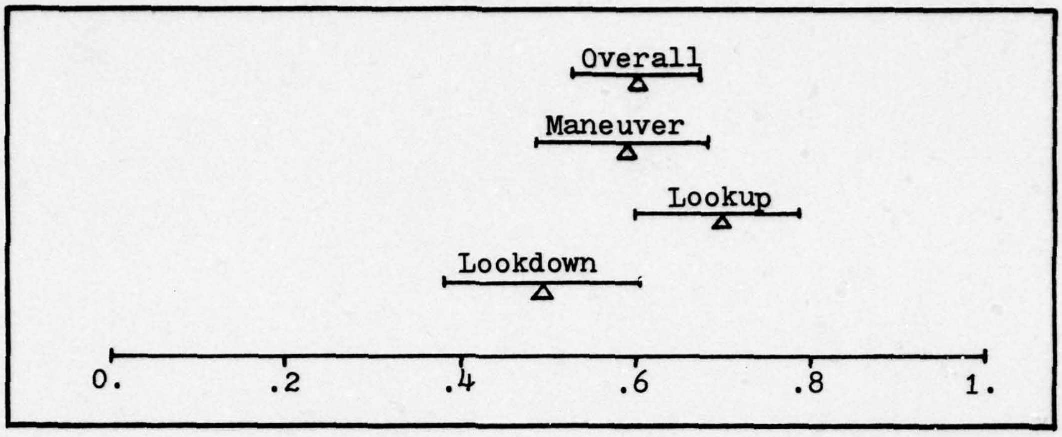


Fig. 4 Means and 80% Confidence Intervals For Proxy Data After Simulated Replication.

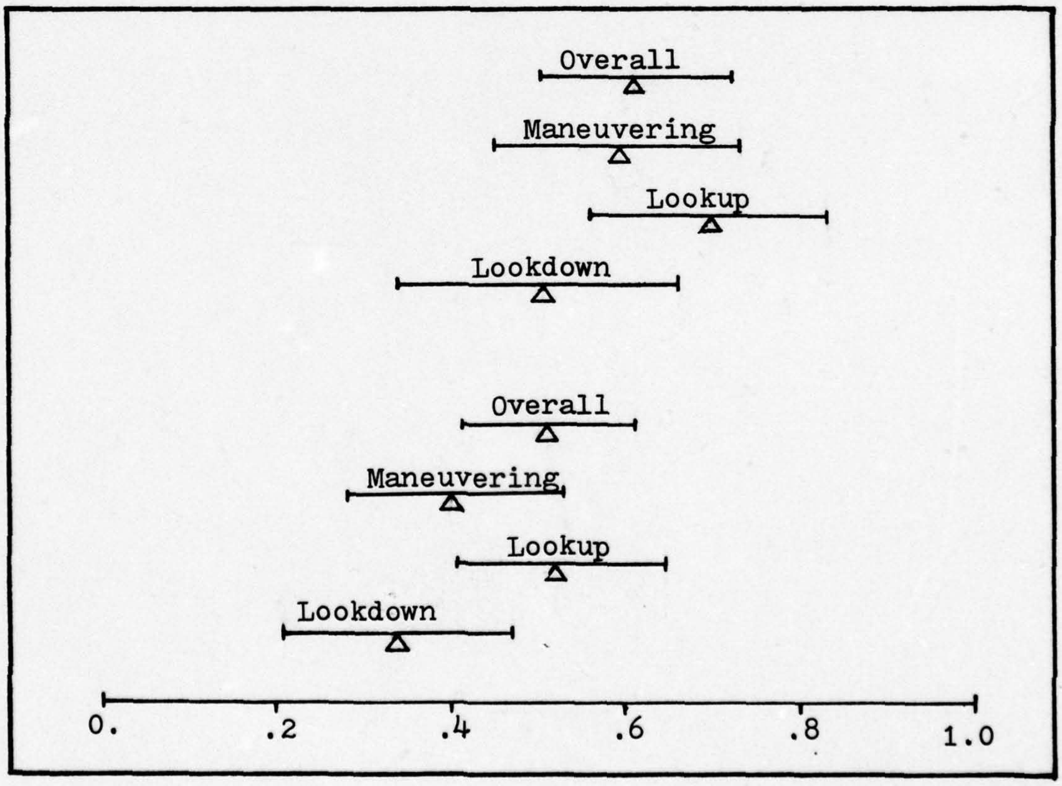


Fig. 5 Means and 80% Confidence Intervals For Proxy Data (Top Set) and Proxy Data Assuming Uniform Prior Distribution (Bottom Set).

using a uniform prior distribution would not be so striking if only one node were so treated. But it must be stated that for applications such as the one at hand, some shift of the overall mean will occur.

The use of a Bayesian technique, as was demonstrated by doubling sample size, illustrates a method in which historical data can be introduced into an ongoing analysis in order to more accurately reflect the total information available for analysis. The same techniques can be used to incorporate information which is not as easily qualifiable into the analysis. This information might be the opinion of an experienced engineer. Such an opinion might be the result of a detailed analysis, and could be extended to include purely intuitive inputs. It could also represent group opinion extracted by means of the Delphi method or a similar approach. An excellent explanation of the Delphi method can be found in The Delphi Method: An Experimental Study of Group Opinion (Ref 1).

It is, of course, the subjective action of the decision maker to include or reject subjective information, but given reliable sources and intelligent use of available information, further refinement of purely analytic analysis can be achieved.

The immediate problem, given diverse data sources, is to translate usable information into a quantitative format. Basically, we have to answer the questions, "What is your best guess of the outcome?" and "How sure of your response are you?". Two approaches appear to be applicable. In one approach the individual whose information is of interest would be asked to estimate some parameters of a prior distribution. Direct estimation of the Beta parameters is probably a poor choice, because the distribution is not generally familiar, but an estimate of the expected mean or of the number of successes out of some arbitrary number of trials would allow us to establish some measure of central tendency for a prior distribution. The second question could be answered by estimating a variance or a standard deviation. The use of such an approach requires a subject who is knowledgeable enough in his area of expertise for his information to be of value, and who is knowledgeable enough about statistical applications to be capable of accurately quantifying his responses. Assuming that statistical applications is an area not widely understood in the depth required for accurate responses, then such an approach would either exclude information because the subject not knowledgeable in statistical procedures could not be questioned, or the approach would introduce errors due to the depth of knowledge of some subjects.

One step in reducing the impact of the transition from nonquantitative information to quantitative information is to present the subject with drawings of a group of density functions and have him select the one he feels best represents the most probable outcome of the test series. The density functions presented to the subject would be representative of several possible outcomes, in terms of central tendency and dispersion. Such an approach was developed by Paul F. Dienemann of the Rand Corporation (Ref 2). Mr. Dienemann's approach was aimed at the estimation of the costs of future systems. A subject was presented with graphs of nine Beta density functions. Three of these distributions had modes at .25; three more at .50; and finally, three at .75. Each of these groups had three degrees of dispersion. From these nine distributions the subject would select the one he felt best represented his subjective estimate of the cost of a system.

Such an approach assumes that the subject has some familiarity with the principles of probability, and that he has a knowledge of the properties of a probability density function. It does not appear to be an unwarranted assumption that individuals associated with the operational test and evaluation process do have this knowledge. Efforts were made to apply this approach to the present problem. However, Dienemann's functions were found to be unsatisfactory for the modeling application discussed in this thesis. Initially it was thought that the reasons for the unsuitability lay in the fact that there was not a large enough selection of functions from which to choose. To meet this shortcoming, a series of functions were developed, with modes at .5, .6, .7, .8, and .9 (mirror images of these functions could be obtained by setting r' equal to (N-r) for M' = 1-M). This approach was also found to be unsatisfactory because there was an unacceptable shift of the mean to a lower value than would have been experienced had the raw data been treated alone. This shift was accounted for by the fact that prior distributions under study were constructed in such a way that the mode (the most probable value, as well as the point where the function reaches a maximum value) was located at the several points of interest. It can be demonstrated that, with the exception of the mean being collocated with the mode at .5, the mean is always between .5 and the mode for the Beta distribution. Further, the distance between these values increases proportionally with the distance from .5. With a shift in the mean towards the central value, this situation becomes very similar to the argument against using a uniform prior distribution to represent an informationless prior state. The result is the same in that the use of a prior distribution whose mode is located at the point of interest will result in a shift of the final model mean to a lower value. The next step was the construction of a series of probability density functions whose means rather than modes lay at the points of interest. For means between .5 and .7, this approach may have some merit. Three potential probability density functions are illustrated in Figures 6, 7, and 8. The means in these figures are denoted by solid lines; the modes, by a tick at the mode. It is interesting to note the progressive shift of the mode of the distributions to higher values as the mean is shifted farther away from .5. As previously indicated, however, prior

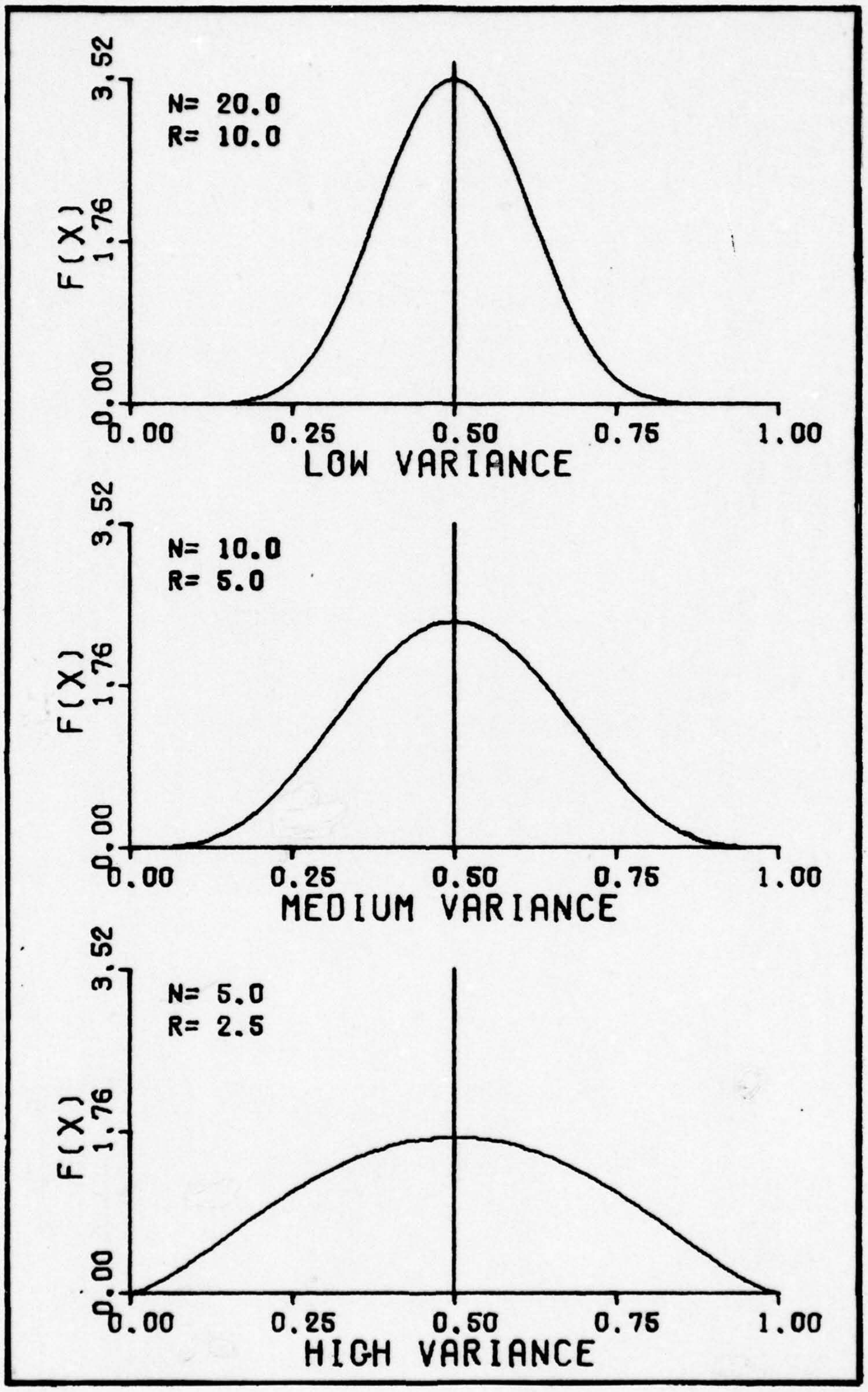


Fig. 6 Subjective Prior Distributions (.5 Mean)

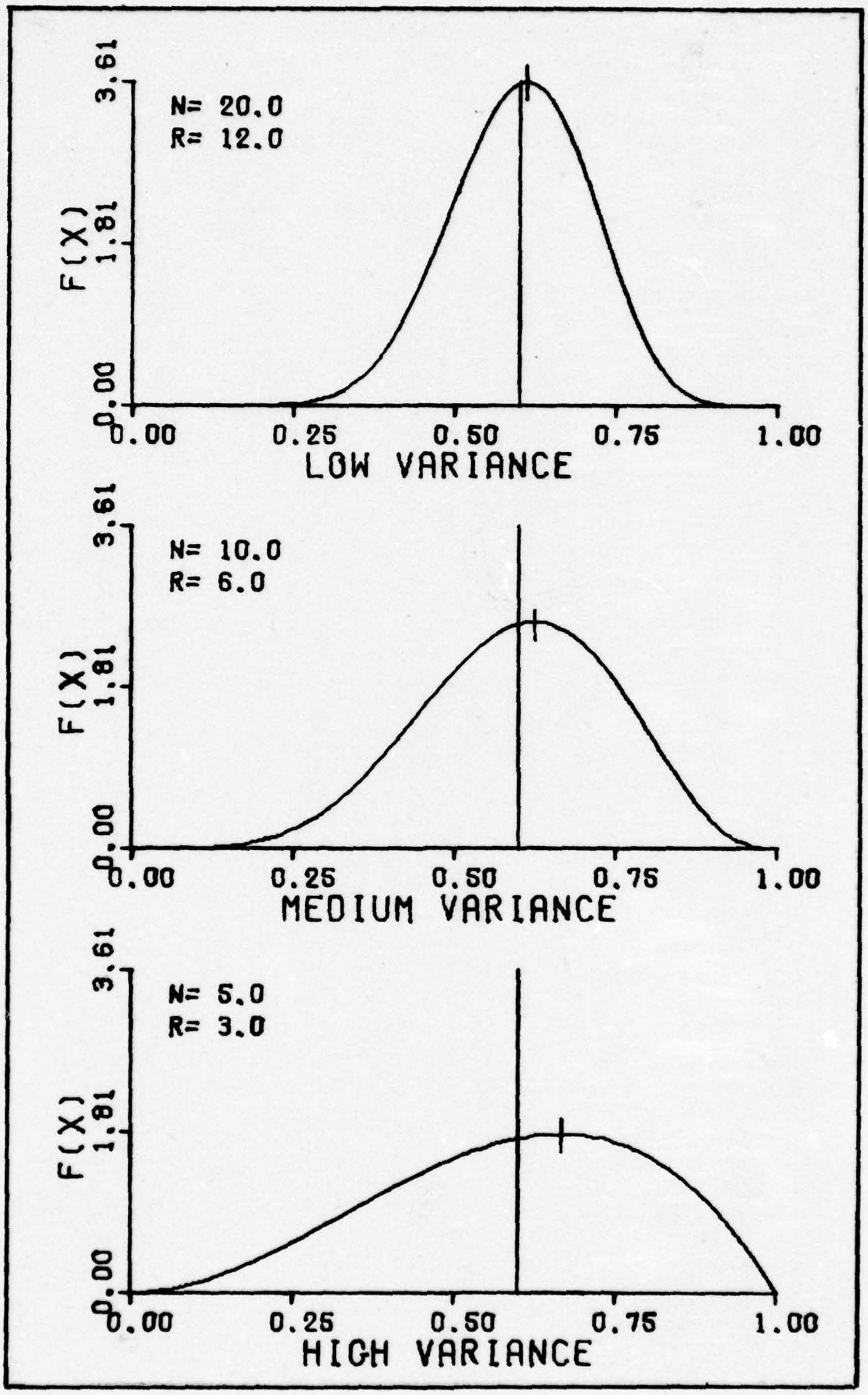


Fig. 7 Subjective Prior Distributions (.6 Mean)

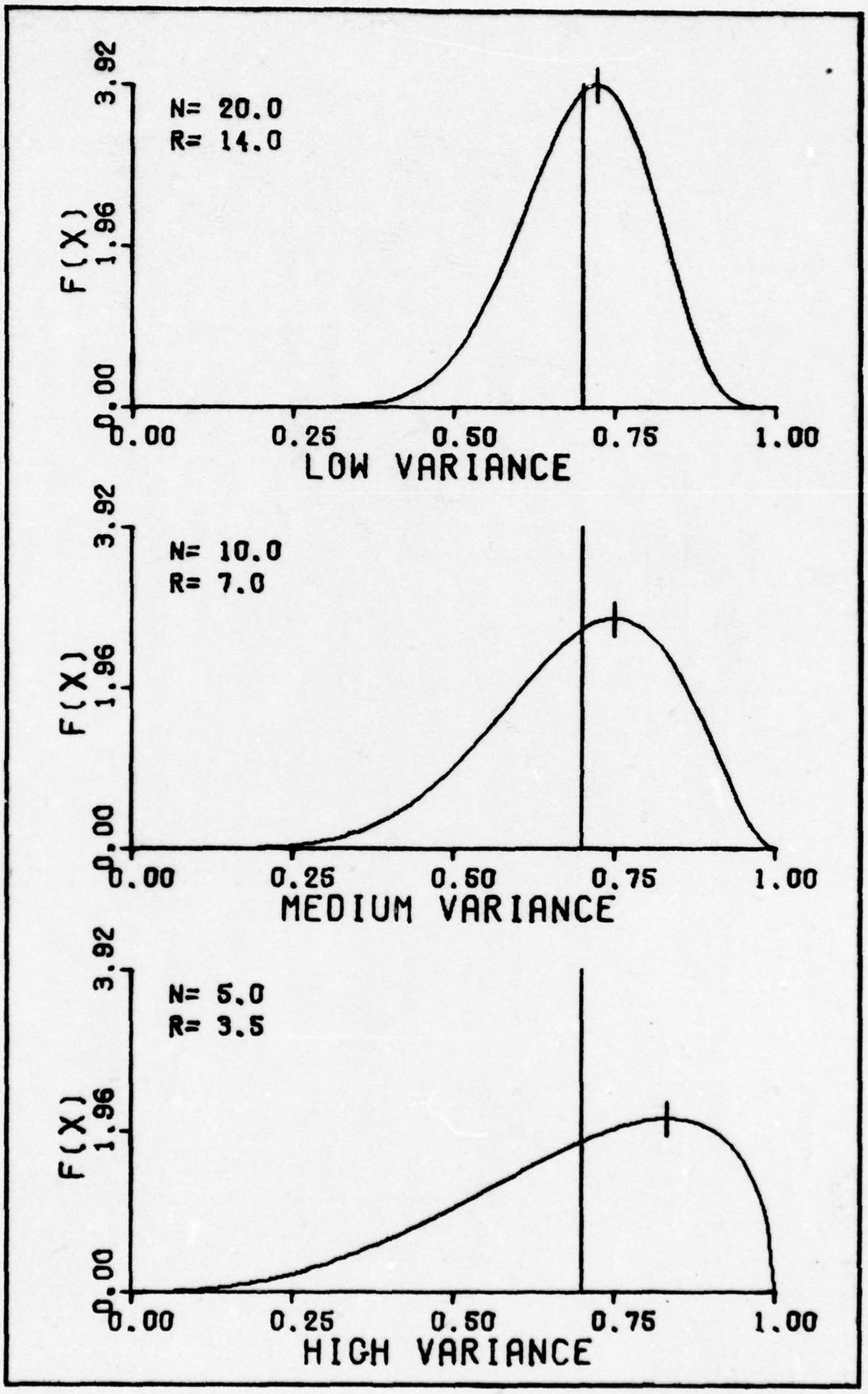


Fig. 8 Subjective Prior Distributions (.7 Mean)

distributions in this region are probably not of significant value in the estimation of the Beta parameters for the individual nodes.

Representations of the density functions with means exceeding .8 are felt to have the most value in this particular application. However, an attempt to arrive at acceptable distributions in this region presented problems. These problems arose because the parameters of the distributions under consideration were held to a low value.

It was felt that the primary utility of the input of subjective information would be in the case where no prior history of a subsystem was available. The nature of the problem requires that the parameters of the sample distribution be relatively small, and it was felt that an approach where the prior would not dominate sample information would be most appropriate. Unfortunately, when the mean of a Beta distribution with relatively small parameters is moved near the end points, the mode approaches either zero or one, and becomes undefined for analytic purposes. Graphically, the distribution becomes "J" shaped. While this is an acceptable probability density function, it does not meet the author's expectations of what a prior distribution "should look like". Figures 9 and 10 illustrate this point. The parameters were varied to determine the point at which the resulting distributions would retain an inverted "U" shape. Using .99 as a maximum acceptable value of the mode leads to the following parameters: $N = 11$, $r = 9.9$. To be able to

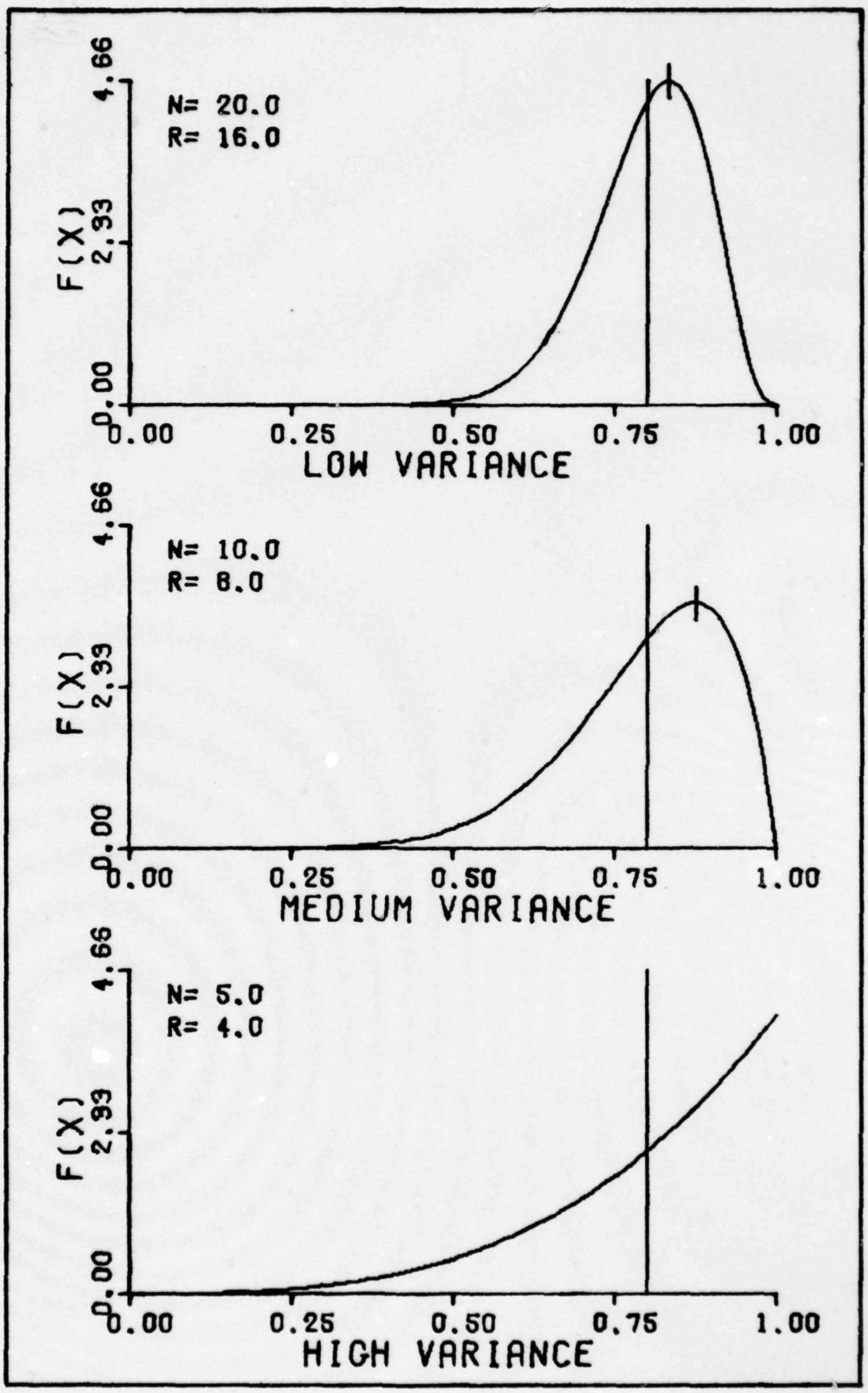


Fig. 9 Subjective Prior Distributions (.8 Mean)

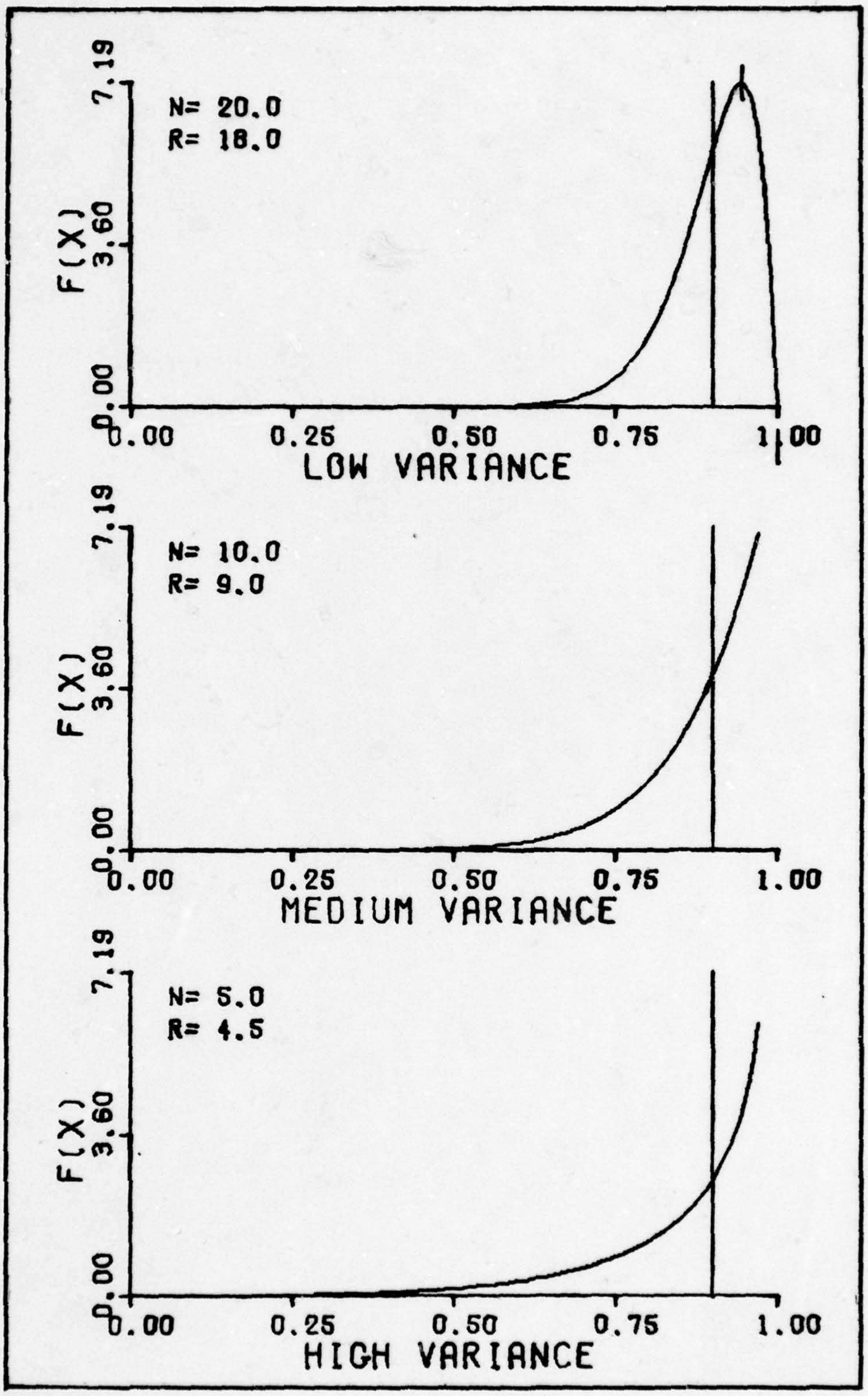


Fig. 10 Subjective Prior Distributions (.9 Mean)

present a variety of dispersions, it is necessary to at least double the "N" parameter and scale the "r" parameter as necessary to keep the mean at .9. Figure 11 illustrates three such density functions with the parameters in the ranges: $11 \leq N \leq 22$, $9.9 \leq r \leq 19.8$.

Neither set of representations of distributions with their means at .9 appears to be acceptable for general use. Reasons for this unacceptability are threefold. First, "J" shaped distributions are not felt to represent an intuitively appealing density function. Secondly, it can be seen that even though the parameters of the distribution representing low variance are double those for the high variance distributions, there is not a readily apparent difference in shape. This is to some degree true of the comparisons between the highest variance function of Figure 9 and the lowest variance functions of Figure 10. While it is true that a greater variety of shapes could be represented, it is felt that the necessary increase in the parameters could have significant impact on the posterior distributions. Finally, there is a strong natural tendency to mentally place the mean of the distributions with means near 1.0 closer to 1.0 than it actually is. Figure 11 illustrates this point. The sharp peaks located at .99, .95, and .94 tend to dominate the illustrations and imply that the mean is closer to the mode than it actually is, or even possibly equal to it. The effect of this last point could easily lead to biasing results in a downward direction in the micro-model

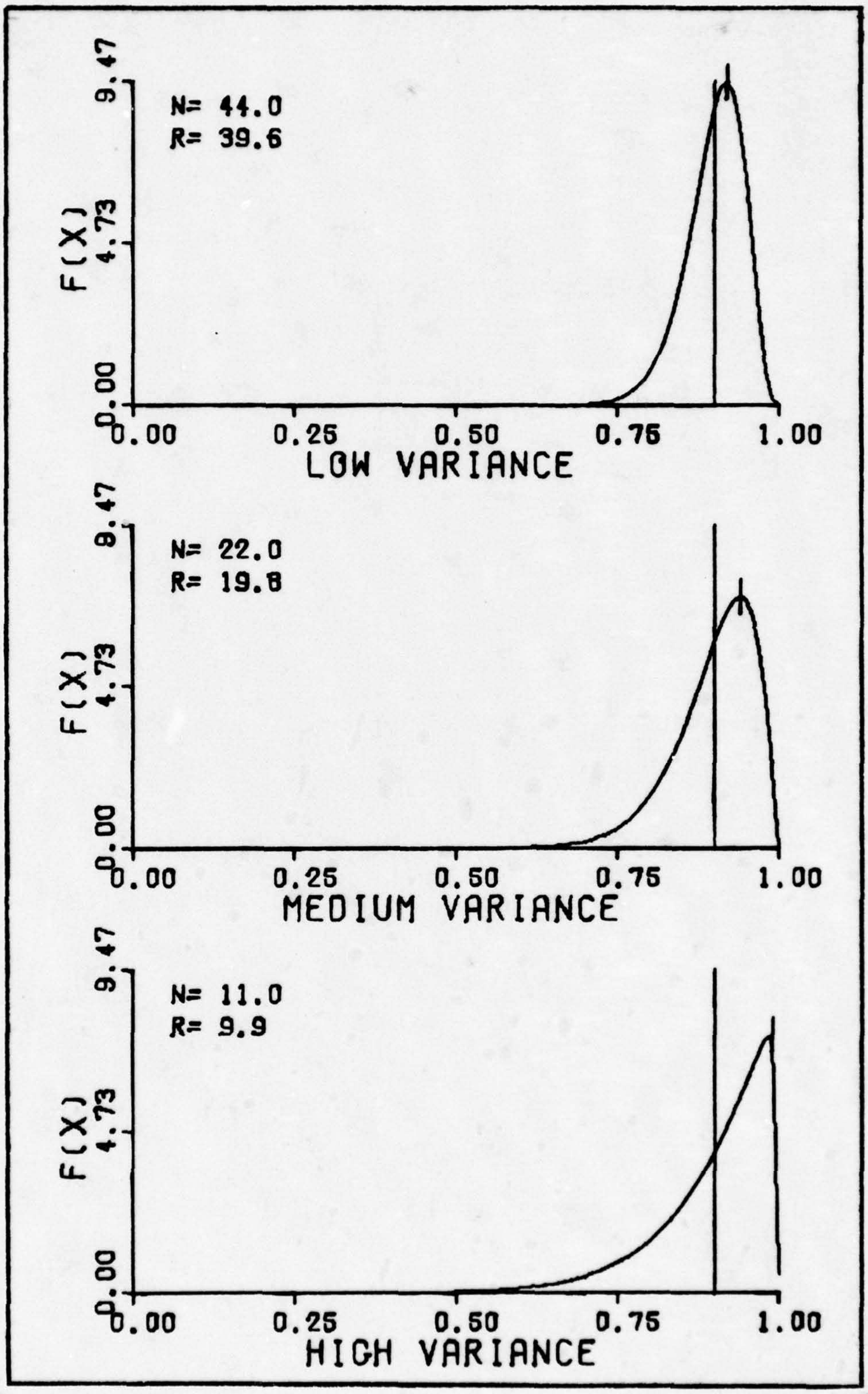


Fig. 11 Subjective Prior Distributions With Increased Parameters (.9 Mean)

In summary, Bayesian techniques appear to have a varying value, depending on the nature of the model selected to represent missile system operation. Because of the questionable validity of historical data, as applied to modeling an entire test program with a single density function, historical input in this context is probably not a viable approach. On the other hand, since the mean of overall system operation can be expected to be located in the central region, subjective input of data may be a valuable approach. If the decision is made to model system operation as a series of nodes, the inclusion of historical data would appear to be appropriate at the nodes representing unmodified subsystems. But, because the means of distributions representing the nodes must achieve a relatively high value, coupled with the author's inability to identify acceptable representative density functions in the region of interest, subjective input of information is questionable in this context.

Sensitivity Analysis

It can be recalled from the illustration given in this chapter's section dealing with Bayesian techniques, that doubling the number of trials decreased the length of the confidence interval of the resulting distribution. With this background, one can intuitively deduce that the variance is inversely proportional to the number of trials. This deduction is generally true, but the variance of a Beta distributed random variable is also a function of its mean as was shown earlier in this chapter by the relationship

$$\sigma^2 = \frac{\mu(1-\mu)}{N+1}$$

It can be seen that the variance is a function of the mean and of the number of trials. The numerator of the above expression reaches a minimum of zero at the 0 and 1 end points and a maximum when the mean equals .5. An increasing number of trials will decrease the variance for a constant mean. This relationship is illustrated in Figure 12.

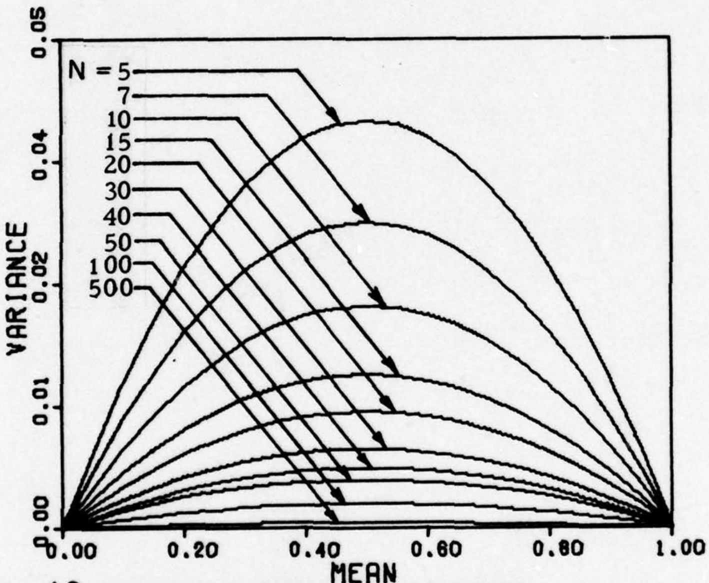


Fig. 12 VARIANCE AS A FUNCTION OF N AND MEAN FOR BETA RANDOM VARIABLES

The length of a confidence interval for any given level of confidence is directly related to the variance, in that an increasing variance will increase the length of an associated confidence interval. This fact, coupled with the relationship of the variance as a function of both the mean and the number of trials, leads to a relationship between the length of a confidence interval at some level of confidence and the mean and number of trials. Figure 13 illustrates this relationship for the Beta distribution and 80% confidence intervals. The figure can be read by laying a straight edge vertically along a mean of interest (bottom scale). The points where the straight edge intersects appropriate values of N determine the limits of the confidence interval and are read from the scale on the left. The figure also can be used for a quick approximation of the results of a test program or for estimations in sensitivity analysis.

It can be seen from Figure 13 that a larger number of trials will tend to reduce the length of the confidence interval. However, in this application, as in many others, there is a limited amount of resources available. Therefore, it can be determined prior to a test program approximately what number of trials will be available for analysis. There also exists a practical range of success rates, since it is improbable that the mean will be very close to 1.0, and it is hoped that it will not be near 0.0. Generally, this will define an envelope of interest which is roughly a parallelogram, since the lines in the central area determining the upper and lower limits of the confidence interval are not severely curved. Examination of a most probable region of results, for example, between .45 and .75 shows the value of increasing the number of tests is greatest for lower initial

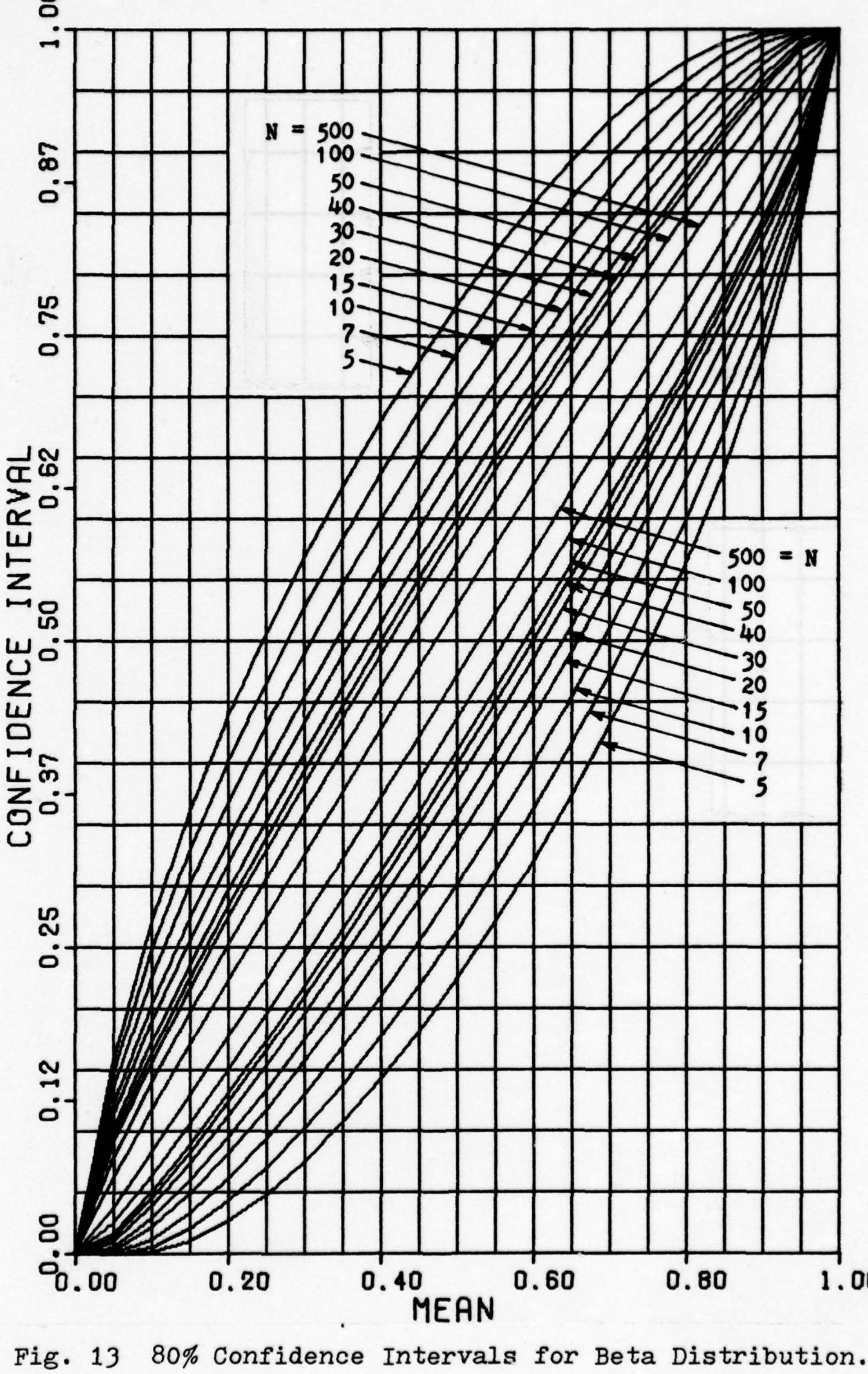


Fig. 13 80% Confidence Intervals for Beta Distribution.

values of N. The effect of the diminishing value of additional tests for purposes of reducing the absolute length of the confidence interval is also illustrated. As an example, increasing the number of tests from 5 to 7 in this area reduces the length of the confidence interval approximately the same amount as increasing the number of tests from 50 to 100, or from 100 to 500. The final length of the confidence interval is not particularly sensitive to the mean in this range, however. Since it can be shown that the length of the confidence interval is most sensitive to the number of tests at small values of the number of tests, it may be profitable to use Figure 13 for planning purposes. In such a capacity, it could be used as an aid to determining the number of missiles to be fired in each mode of operation. For example, a particular mode may be of interest in a particular evaluation, and it may be desired to acquire as much information as possible about this mode. Reference to Figure 13 will indicate the minimum number of firings required to gain acceptable test results in terms of length of confidence interval for other modes. All remaining resources can then be devoted to the particular mode of interest. In such situations it may be wise to withhold some missiles to allow for "no test" firings, since losing one or more data points in the areas planned around minimum requirements may have significant impact on test results. Those missiles withheld can be allocated as desired during the test program to arrive at acceptable results.

IV. Conclusions

This study has been aimed primarily at the identification of a method of data reduction in the specific case of the evaluation of tactical air-to-air guided missile systems. The objective has been to identify a method of estimating the probability of kill in the three modes of operation --- look-up, look-down, and against a target maneuvering in excess of 4 g's; and to estimate the confidence interval about these probabilities of kill which do not exceed .20 in length at the 80 percent level of confidence.

The first method explored in meeting this objective was regression analysis. The approach was not found to offer a suitable solution to the problem at hand. The shortcomings of the approach in meeting the objective may have been partially due to the nature of the data. However, the large number of independent variables which are recorded, and the other variables which are not recorded but may have significant impact on the system's operation, are felt to make the approach of questionable value in this application. If those variables which have significant impact on system operation could be identified, it is felt that this approach would be valuable in predicting system operation. This study did not identify those variables.

Next, mathematical modeling was explored, and a model representing system operation was defined. This model was analyzed using Monte Carlo and analytic techniques. The analytic approach was accurate and simple enough in handling, that it was used in preference to the Monte Carlo method. However, the results of the Monte Carlo simulation were valuable in validating the analytic model, and may be the only feasible approach in more complicated models.

In the analytic approach, each of the nodes of the previously defined model was modeled as a Beta density function. Relationships were developed to determine the mean and variance of a Beta distribution, which was used to represent overall system operation. Inferences were drawn, based upon both this distribution as well as a single distribution whose parameters were the number of successful missions and the total number of launches. Findings of this approach indicated that the pure analytic approach did not meet the objectives. Specifically, even though a mean was obtained for each mode of operation, there was overlap of the confidence intervals regardless of the mode of operation or the method of reduction. Further, it was noted that the lengths of the confidence intervals were in excess of .2 in all cases, ranging in length from .26 to .32. Reference to Figure 13 in Chapter 3 illustrates that with an expected mean in the range of .50 to .70, the length of the resulting confidence interval is relatively insensitive to the actual value of the mean. The length of the confidence interval is primarily sensitive to the number of trials. To achieve a confidence interval of .20 length requires at least 30 trials throughout the range, and the number approaches 40 trials at a mean of .5. Analysis of the combined results of all 34 launches further illustrates this point with its confidence interval of .21 length in both methods of reduction.

Because of the above-listed shortcomings and the known limited number of launches, this method appears to have suitability only in the inference from overall test results.

Finally, Bayesian techniques were examined as a method for reducing the lengths of the confidence intervals. Such methods permit the inclusion of data from either historical or subjective sources. To demonstrate the inclusion of historical data, the proxy test results were assumed to have been replicated and then re-analyzed. Since this would constitute a complete replication of the test series, it may be a best case situation in that the values of the parameters describing all nodes would be doubled. On the other hand, it does not recognize historical data for more than one previous test of any subsystem. Results for this analysis were encouraging, in that the confidence intervals for two modes of operation met the objective of .20 in length. The one case which did not meet this objective was centrally located (with a mean at .50), and had the fewest number of trials. Both of these factors tended to increase the length of the interval, but the resulting length was still only .23. It was also noted that there was only a .01 overlap in confidence intervals for the look-up and look-down modes of operation. Since these are the only two mutually exclusive modes, it is improbable that conditions would arise where the means for all modes could be statistically separated. Further, it is unknown how realistic the .18 spread between the means of the proxy data actually is. If this spread is larger than that encountered in practice, then the problem of statistical separation would be compounded. On the other hand, if this is a smaller spread than that which would normally be expected, then the inclusion of historical data to update the nodes of the model may be a profitable approach.

Because of the expected incremental changes in a missile system, the application of historical update methods to a model representing only the number of successes and number of trials is questionable. However, in this case, the inclusion of subjective data appears to be practical. Graphs of appropriate probability density functions were presented in Figures 6, 7, and 8 in Chapter 3, and the use of these or similar functions appears to be a good medium for quantifying subjective information. For the case of the model of missile system operation consisting of a series of nodes, it is questionable whether subjective information can be accurately quantified. This is because the requirement to present an acceptable density function with a mean equal to or greater than about .80 could not be met. It is necessary to be able to present distributions in this region to allow the subject to select from a variety of distributions in the most probable area of successful operation. As indicated earlier, each subsystem of the missile must function successfully approximately 89 percent of the time to arrive at a .50 overall probability of success. It was felt that the distributions represented in Figures 9, 10, and 11 would lead a subject to select a distribution with a mean lower than his true estimation of the mean.

Based on the above reasons, it is concluded that, of the two Bayesian techniques explored, each is applicable to only one type of model under consideration. That is, subjective input of data may be valuable for the consideration of the macro-type of model, and historical update is most appropriate for the micro-model.

In summary, methods have been explored to attempt to statistically separate the expected probabilities of kill for various modes of operation. It is improbable that the number of missiles available would be sufficient to allow such a separation in a purely analytic model. However, the inclusion of data from other sources presents a vehicle by which to refine the results obtainable from a purely analytic model. Depending on the availability of data and the type of model chosen, it is possible that the combination of an analytic approach and Bayesian techniques may meet the earlier stated objective. However, it is highly improbable that this approach (or any other) could permit the statistical separation of the means in all situations.

Recommendations

While the methods proposed in this study are not guaranteed to meet the stated objectives, it is felt that the approaches are technically sound and may be of value in a more limited scope than was originally envisioned. Specifically, the determination of an acceptable confidence interval about the overall probability of kill for the system represents an improvement over current methods of data reduction. It is also probable that the techniques will be usable in some situations to infer probabilities of kill, as well as confidence intervals for the modes of operation. It is, therefore, recommended that the techniques included in this paper be considered by AFTEC to support their data reduction efforts.

Secondly, it is recommended that further research be conducted in the area of developing adequate representations of distributions whose mean is near the endpoint of a closed range. These representations would be presented to a subject to allow him to quantify his subjective opinion of the outcome of a test series. An objective of such an effort should be to accurately extract that information, and the perceptions of subjects should be recognized. It is felt that the domination of the mean by the mode of the graphs presented in Chapter three would make those functions of questionable value in extracting subjective information. Possible areas of research could include the artificial decrease of the parameters of a Beta distribution. High values of parameters could be used to shape the function, but these could be reduced for the physical update process so that the severe impact of the true shape parameters would be brought more closely into balance with the sample data. Such an approach may not be technically proper, but as noted earlier, " . . . the ultimate aim of the Bayesian is to use the posterior distribution in an inferential or decision making process" (Ref 4:486). For this reason, it is necessary to accurately extract available information. Towards this end, adjustments may be justified on the basis of the perceptions of subjects.

In the same area, an appropriate research area would be the application of closely related families of distributions to represent prior subjective information. Other families of curves may adequately represent the type of information addressed above, and a transformation of parameters or truncation of distributions may be possible to adequately quantify the information.

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APPENDIX A

Proxy Data

This appendix contains a description of the method used to generate the proxy data which was analyzed in the body of this thesis. A listing of the data is also included in Table 5. All data was randomly generated from possible launch conditions as determined from unclassified sources. Thirty four launches were chosen for analysis because this number appeared to be reasonably representative of a test series. The 34 launches were divided into four separate categories which were further subdivided to simulate a test plan which would test the missile in all flight regimes. The four major categories were designed to test the missile guidance function (9 missiles), the proximity fuzing function (8 missiles), the warhead lethality (8 missiles), and overall missile system effectiveness (9 missiles). Other functions which were represented by nodes in the micro-model which was developed in the body of the thesis were considered to be randomly distributed throughout the test program. For example, it was assumed to be equiprobable that a launch, launch sequence, or contact fuze failure would occur on any launch. On the other hand, the probabilities of successful guidance, proximity fuzing, and warhead lethality were felt to be under more direct control of the test planners and these probabilities could be altered by the selection of the launch parameters and encounter geometry. Two levels of the probability of success of these nodes were chosen to represent that subsystem's operation. It was reasoned that if a particular subsystem was of primary interest in a given launch, this launch would be conducted under circumstances which would more severely test that subsystem. But the stress on subsystems whose primary function occurred chronologically before the primary phase of interest would be relaxed in order to increase the probability of reaching the phase of interest. To equalize the effect of altering the probability of successful operation, all subsystems were given the higher probability of success regardless of their chronological position in the model if the launch was not designed to test that subsystem. The means of the probabilities of successful missile system operation at the individual nodes were set at the following arbitrary levels:

Launch	.95
Launch Sequence	.90
Guidance	
high	.80
low	.60
No Premature Proximity Fuzing	.90
Impact	.40
Contact Fuze Operation	.98
Proximity Fuze Operation	
high	.85
low	.65
Warhead Lethality	
high	.95
low	.75

Subdivision of the main categories was accomplished to predetermine an altitude profile and to insure an appropriate Look-up (LU)/Look-down (LD) mix. The launch altitude blocks considered were low (surface to 10,000 feet), medium (2,000 to 30,000 feet), and high (35,000 to 50,000 feet). Target altitude blocks were identical except for the high altitude block where the target had a maximum altitude of 75,000 feet. All launches except the combined system launches in the high and low altitude blocks were predetermined to be in the LU or LD mode. The remaining launches including the combined system launches had randomly distributed modes of operation based on randomly selected launch and target altitudes (RND).

Table 4 presents a more detailed enumeration of the predetermined launch conditions where P_g refers to the probability of successful guidance, P_{pp} refers to the probability of no premature proximity fuze operation, and P_{le} refers to the probability that the warhead is lethal.

Table IV
Predetermined Launch Conditions

Test	Total Number	Number	Alt. Block	Mode	P_g		P_{pp}		P_{le}	
					High	Low	High	Low	High	Low
Guidance	9	6	Low	LD		x	x		x	
		3	High	LU		x	x		x	
Fuzing	8	5	Med	RND	x			x	x	
		3	High	LU	x			x	x	
Lethality	8	5	Med	RND	x		x			x
		3	High	LU	x		x			x
Combined System	9	4	Med	RND		x	x		x	
		5	Med	RND	x			x	x	

The airspeed of the launch vehicle was randomly generated from a uniform distribution of representative airspeeds within the blocks. The ranges of airspeeds were .6 mach (m) to 1.5m at low altitude, .75m to 2.0m in the medium altitude block, and .85m to 2.5m in the high altitude block. Target airspeeds were randomly generated from a normal distribution with a mean of the launch platform mach and a standard deviation of .25m.

Slant ranges were randomly generated from uniform distributions which varied with the altitude. The ranges were from 2 to 8 nautical miles (nm) in the low altitude block, 5 to 17 nm in the medium altitude block, and from 5 to 28 nm in the high altitude block.

All high altitude launches were assumed to be against non-maneuvering targets. For the lower two altitude blocks the target was randomly determined to be either maneuvering or non-maneuvering. Target g loading was then randomly selected from a uniform distribution between 1 and 4 g's for a non-maneuvering target, and between 4 and 7 g's for a maneuvering target.

The aspect angles were generated from a normal distribution with a mean of zero and a standard deviation of 25 degrees. One hundred and eighty degrees were added to all negative numbers. This procedure was selected to concentrate the launches about the head-on and tail-on aspects while still giving a reasonably wide dispersion to the beam aspect encounters.

After the launch parameters had been determined, the results of each launch were determined by generating random numbers from a uniform (0,1) distribution. One random number was generated for each node in the mission of each missile. If the random number representing the operation of the missile at a node exceeded the earlier defined mean probability of success applicable to that node, the missile was assumed to have failed at that point in its mission. For example, a missile which was fired to test the guidance system would have a .60 probability of successful guidance. If the random number associated with the guidance node of this missile exceeded .60, the missile would have been assigned a failure due to guidance. If the random number were equal to or less than .60 the missile would be assumed to have guided successfully.

The results of this procedure are presented in Table 5 which follows.

Table V
Summary of Launch Conditions and Missile Failures

Launch Conditions								Failures									
Launch Altitude (x1000')	Target Altitude (x1000')	Slant Range	Target G-Load	Launch G-Load	Aspect Angle	Launch Mach	Target Mach	Launch	Launch Sequence	Guidance	Premature Fuze	Impact	Proximity Fuze	Contact Fuze	W/H Lethality	Overall Failures	
4.7	2.1	6.0	5.6	2.3	168	.78	.71										Guidance
4.9	1.3	7.9	4.3	1.1	14	.96	1.02										
2.0	0.4	5.9	6.6	2.6	140	1.05	.62		x							x	
2.7	0.9	3.9	5.4	1.6	26	1.08	.65					x					
7.5	6.3	5.9	5.8	1.8	20	.85	1.12										
9.2	3.5	3.6	2.9	2.1	173	1.11	1.58					x					
47.9	64.6	16.8	1.4	1.6	19	1.31	1.06					x					
40.2	50.7	18.9	2.1	1.0	20	1.64	1.13					x	x				
41.5	60.9	17.0	3.1	1.0	169	2.45	2.54								x	x	
13.5	22.2	9.4	1.4	2.9	3	.89	.73										Fuzing
18.2	7.1	6.3	1.7	3.0	166	1.44	1.41					x	x		x	x	
2.1	9.6	15.3	4.9	2.8	15	1.13	1.10										
8.8	9.1	11.5	4.5	1.7	25	1.12	1.06										
7.7	4.9	15.1	5.9	1.4	13	1.84	1.99			x						x	
45.3	50.1	22.4	2.1	3.6	4	1.91	1.84								x	x	
50.0	70.3	16.2	2.6	3.3	22	1.26	1.57										
49.3	73.0	16.3	2.9	1.7	177	1.47	1.46										
24.5	15.6	6.3	1.2	5.2	159	1.54	1.47	x								x	Warhead Lethality
15.0	13.6	5.5	6.6	1.2	11	.81	.86				x					x	
8.4	4.4	7.9	5.5	1.1	5	1.57	1.51			x						x	
27.4	23.2	13.6	5.4	1.9	159	1.78	1.54					x					
18.0	12.1	16.4	6.3	2.4	17	.98	1.19	x								x	
42.3	50.9	15.3	1.8	2.9	2	1.12	1.50										
38.9	43.6	11.0	1.2	2.9	152	.95	1.23										
41.4	66.0	16.2	1.2	2.8	33	1.17	1.20			x						x	
18.0	23.6	12.3	6.6	1.7	163	.76	.67										Combined Systems
28.5	23.8	7.0	5.8	1.7	177	1.22	1.15					x					
4.5	5.7	16.3	6.6	2.3	178	.91	.76					x		x			
15.0	27.5	5.5	1.1	3.8	132	.78	.87										
12.6	4.1	16.6	5.6	4.8	125	1.13	.90										
3.9	6.2	5.1	4.6	3.6	29	.90	.96				x					x	
11.5	11.6	14.1	4.9	2.1	150	.76	.58					x					
2.6	11.8	12.3	6.3	1.7	1	1.83	2.05		x							x	
14.0	29.3	5.5	2.6	2.9	13	1.24	1.08										

APPENDIX B

Program to Evaluate Parameters of Model Beta Distribution

This appendix includes a listing of and instructions for the use of a program to evaluate the mean, variance, N, r, and the resulting confidence intervals for the model proposed in Chapter 2 of this thesis. The program was written for a Texas Instruments Programmable-58 calculator (TI-58). Due to its length, it would probably be more convenient to use a TI Programmable-59 (TI-59) calculator and transfer the program to a magnetic card in order to avoid the requirement to manually enter the program for each use.

The format of a hand-held calculator routine was chosen because the program is not particularly complicated and it bypasses the turn-around time requirement associated with the use of a large scale digital computer. Also this format has the advantage of mobility, a feature judged to be desirable by personnel of AFTEC for use in field tests and on-the-spot sensitivity analyses.

The listing of the program is contained in Figure 14. The program listed will evaluate the micro-model for five series nodes prior to the impact/fuzing loop. The extra node was entered to accommodate launch platform/missile system evaluation as well as a missile system evaluation. The extra node can be bypassed by entering "1" for both the number of success and trials at the first node. The program can be modified to accommodate any number of series nodes directly by changing program step 111 to the desired number of series nodes prior to the impact/fuzing loop.

To enter the program into the TI-58, the calculator must be repartitioned to accommodate 320 program steps and 20 memory locations. The TI-59 does not require repartitioning. After the program has been entered the calculator is initialized by pressing "2nd, B'". The display is the number of series nodes prior to the impact/fuzing loop. Test results are then entered as the number of successes and the number of trials for each node. Each entry is followed by a "R/S" command. The pairs of entries should be made in the following order:

Launch platform (optional--bypass by entering "1" and "1")
Missile launch
Launch sequence
Guidance
No premature proximity fuzing
Target impact
Contact fuze operation
Proximity fuze operation
Warhead lethality

The display after the eighteenth entry is the variance of the final distribution model. Successive "R/S" commands will result in displays of the mean, the parameters N and r, and finally in approximation of the Beta coefficient to be used for calculating confidence intervals.

The approximation of the Beta coefficient is obtained by evaluating the Beta function with the values of N and r already computed and with a coefficient of one in a Simpson's rule approximation. The reciprocal of the result is then used as the coefficient in subsequent calculations. The accuracy of this approximation is a function of the number of increments used to evaluate the function. This program uses 20 increments. Increased accuracy can be obtained by increasing the number of increments (program steps 304 and 305) with an accompanying increase in the time required for the computation, or by direct calculation of the required Gamma functions. Such direct computations of the Gamma functions and the Beta coefficient for the three modes of operation and for the overall results of the basic data presented in Chapter three and for the same four cases with each node being replicated did not result in an error greater than .0001 (.01%) for any of the eight cases.

This is the end of the formal program. However, confidence intervals can be calculated by using the Simpson's rule Master library routine. At this point the calculator has the Beta coefficient, (r-1), and (N-r-1) stored in locations used by the Simpson rule subroutine (program steps 000 through 021). The confidence intervals can be estimated by determining those points at which .10 of the total area under the function is excluded from each side of the function. Accurate initial estimates of the limits can be determined from Figure 15 in Appendix 3. If the integral of the function is approximated from 0.0 to the estimated lower limit and from the estimated upper limit to 1.0, the central portion of the function is avoided and six to eight increments provide an accurate estimate of the area excluded.

This program will not operate properly in any of the following cases.

1. All proximity fuzes function properly:
2. All missiles impact the target and all contact fuzes function properly:
3. All proximity fuzes and contact fuzes fail.

Each of these conditions results in a mean of either 1.0 or 0.0 for the impact/fuzing loop and 0.0 variance. The parameter N for the loop is computed from the following equation:

$$N = \frac{\mu\ (1-\mu)}{\sigma^2} - 1$$

The required division of zero by zero results in an error condition in the calculator. The results of cases one and two can be calculated from the relationships developed in Chapter two. Case three will result in a 0.0 mean and a 0.0 variance for the system.

TITLE Evaluate Parameters of Model Beta Dist PAGE 1 OF 3 **TI Programmable**

PROGRAMMER Vogel DATE Jan 69 **Program Record**

Partitioning (Op 17) 3 1 9 . 1 9 Library Module MASTER Printer ______ Cards ______

PROGRAM DESCRIPTION

Compute σ^2, μ, N, r, and an estimate of the Beta coefficient for a Beta distribution representing the probability of successful missile system operation using the model proposed in Chapter I and equations developed in Chapter Two.

Display values should not be altered during program execution because some values are used in subsequent operations.

USER INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS			DISPLAY
1	Initialize	0		2ND	B'	5.
2	Enter r_i and N_i in pairs in the following order: Launch Platform, Launch, Launch Sequence, Guidance, No Premature Proximity Fuzing, Impact, Contact Fuzing, Proximity Fuzing, Warhead Lethality.	r_1			R/S	
		N_1			R/S	
		⋮			⋮	
	Display after ninth pair of entries is variance	r_9			R/S	
		N_9			R/S	Variance
3	Display Mean				R/S	Mean
4	Display N				R/S	N
5	Display r				R/S	r
6	Display Beta Coefficient				R/S	Beta Coefficient
7	Confidence Intervals (CI) can be estimated. N-r-1, r-1, and the Beta Coefficient are stored in locations used by the Simpson Rule Approximation. "0" is stored as the lower limit. "1" is stored as the upper limit.		2ND	Pgm	09	
		Estimated Lower Limit of CI			B	
		Number of Increments			C	
					D	Area Excluded

USER DEFINED KEYS	
A, B, C, D, E, A'	Reserved for Simpson Rule Routine
B'	Initialize
C'	
D'	
E'	

DATA REGISTERS (INV) 0–9: (blank)

LABELS (Op 08): INV X, lnx X, CE, CLR X, x⇄t, x², √x, 1/x, STO X, RCL X, SUM X, y^x X, EE X, (,), ÷, GTO, ×, SBR, −, RST, +, R/S, ., +/−, =, ...

FLAGS 0 1 2 3 4 5 6 7 8 9

 1014956-1

Fig. 14 Listing of Program to Determine Parameters for Model Beta Distribution

TITLE *Evaluate Parameters of Model Beta Dist.* PAGE *2* OF *3*

PROGRAMMER *Vogel* DATE *Jan 79*

TI Programmable Coding Form

LOC	CODE	KEY	COMMENTS
00	76	Lbl	Subroutine for Simpson Rule Approximation of Beta Function
	16	A'	
	42	STO	
	09	09	
	45	Y^x	
	43	RCL	
	06	06	
	65	X	
	53	(	
	01	1	
01	75	−	
	43	RCL	
	09	09	
	54	)	
	45	Y^x	
	43	RCL	
	07	07	
	65	X	
	43	RCL	
	08	08	
02	54	)	
	92	INV/SBR	
	76	Lbl	Set Loop Counters
	25	CLR	
	04	4	
	42	STO	
	00	00	
	01	1	
	04	4	
	42	STO	
03	08	08	
	92	INV/SBR	
	76	Lbl	Compute Variance for n series Beta Distributions
	45	Y^x	
	43	RCL	
	07	07	
	42	STO	
	09	09	
	53	(	
	01	1	
04	76	Lbl	
	22	INV	
	69	OP	
	38	38	
	65	X	
	73	RCL/Ind	
	08	08	
	97	Dsz	
	07	7	
	22	INV	
05	75	−	
	53	(	
	01	1	
	76	Lbl	
	23	ln x	

LOC	CODE	KEY	COMMENTS
	69	OP	
	30	30	
	65	X	
	73	RCL/Ind	
	00	00	
06	97	Dsz	
	09	9	
	23	ln x	
	54	)	
	33	x^2	
	54	)	
	92	INV/SBR	
	76	Lbl	Load Loop- Takes r and N Computes: μ, $\mu\left(\frac{r+1}{N+1}\right)$
	44	SUM	
	91	R/S	
07	64	Prd/Ind	
	00	00	
	65	X	
	53	(	
	24	CE	
	85	+	
	01	1	
	54	)	
	55	÷	
	53	(	
08	91	R/S	
	22	INV	
	64	Prd/Ind	
	00	00	
	65	X	
	53	(	
	24	CE	
	85	+	
	01	1	
	54	)	
09	54	)	
	54	)	
	64	Prd/Ind	
	08	08	
	92	INV/SBR	
	76	Lbl	Initialize
	17	B'	
	01	1	
	04	4	
	42	STO	
10	00	00	
	01	1	
	76	Lbl	
	42	STO	
	72	STO/Ind	
	00	00	
	97	Dsz	
	00	0	
	42	STO	
	71	SBR	

LOC	CODE	KEY	COMMENTS
11	25	CLR	Set up Storage for Load Loop
	05	5 *	* Note: Step 111 Controls Number of Series Nodes.
	42	STO	
	09	09	
	76	Lbl	
	43	RCL	
	71	SBR	
	44	SUM	
	97	Dsz	
	09	9	
12	43	RCL	
	69	OP	
	30	30	
	76	Lbl	
	52	EE	
	69	OP	
	38	38	
	71	SBR	
	44	SUM	
	97	Dsz	
13	00	0	
	52	EE	
	71	SBR	
	25	CLR	
	71	SBR	
	44	SUM	
	69	OP	Compute σ^2 Loop
	37	37	
	71	SBR	
	25	CLR	
14	71	SBR	
	45	Y^x	
	42	STO	
	05	05	
	65	X	
	71	SBR	
	45	Y^x	
	42	STO	
	06	06	
	85	÷	
15	43	RCL	
	05	05	
	65	X	
	43	RCL	
	02	02	
	42	STO	
	05	05	
	33	x^2	
	85	+	
	43	RCL	

MERGED CODES

62	72 STO	83 GTO
63	73 RCL	84
64	74 SUM	92 INV SBR

TEXAS INSTRUMENTS INCORPORATED

Fig. 14 Continued

TITLE Evaluate Parameters of Mode Beta Dist PAGE 3 OF 3

PROGRAMMER Vogel DATE Jan 79

TI Programmable Coding Form

LOC	CODE	KEY	COMMENTS
16	03	03	
1	49	Prd	
2	05	05	
3	33	x^2	
4	65	X	
5	43	RCL	
6	06	06	
7	54	)	
8	65	X	
9	43	RCL	
17	01	01	
1	85	+	
2	71	SBR	
3	45	Y^x	
4	42	STO	
5	15	15	
6	65	X	
7	43	RCL	
8	05	05	
9	54	)	Compute σ^2_{Loop}
18 0	65	X	
1	02	2	
2	94	+/-	
3	54	)	
4	44	SUM	
5	15	15	
6	71	SBR	
7	25	CLR	
8	03	3	
9	42	STO	
19 0	07	07	
1	71	SBR	
2	45	Y^x	
3	44	SUM	
4	15	15	
5	71	SBR	
6	25	CLR	
7	02	2	
8	42	STO	
9	07	07	
20 0	71	SBR	
1	45	Y^x	
2	44	SUM	
3	15	15	
4	43	RCL	Compute μ_{Loop}
5	05	05	
6	75	−	
7	53	(	
8	24	CE	
9	65	X	
21 0	43	RCL	
1	01	01	
2	75	−	
3	43	RCL	
4	01	01	

LOC	CODE	KEY	COMMENTS
5	54	)	
6	54	)	
7	42	STO	
8	06	06	
9	49	Prd	Compute N_{Loop}
22	04	04	
1	65	X	
2	53	(	
3	24	CE	
4	94	+/-	
5	85	+	
6	01	1	
7	54	)	
8	55	÷	
9	43	RCL	
23 0	15	15	
1	75	−	
2	01	1	
3	54	)	
4	42	STO	Compute $N_L(N_L+1)$
5	05	05	
6	85	+	
7	33	x^2	
8	54	)	
9	22	INV	
24 0	49	Prd	
1	14	14	Compute r_{Loop}
2	43	RCL	
3	05	05	
4	65	X	
5	43	RCL	
6	06	06	
7	54	)	
8	85	+	Compute $\Pi \frac{r(r+1)}{N(N+1)}$
9	33	x^2	
25 0	54	)	
1	49	Prd	
2	14	14	
3	71	SBR	Compute σ^2_{TOTAL}
4	25	CLR	
5	69	OP	
6	20	20	
7	69	OP	
8	28	28	
9	71	SBR	
26 0	45	Y^x	
1	91	R/S	
2	42	STO	Display μ_{TOTAL}
3	05	05	
4	43	RCL	
5	04	04	
6	91	R/S	
7	65	X	
8	53	(	
9	24	CE	

LOC	CODE	KEY	COMMENTS
27	94	+/-	Compute N_{TOTAL}
1	85	+	
2	01	1	
3	54	)	
4	55	÷	
5	43	RCL	
6	05	05	
7	75	−	
8	01	1	
9	42	STO	
28	08	08	
1	42	STO	
2	02	02	
3	54	)	
4	42	STO	
5	07	07	
6	91	R/S	
7	65	X	Compute r_{TOTAL}
8	43	RCL	
9	04	04	
29 0	54	)	
1	91	R/S	
2	42	STO	Compute Beta Coefficient
3	06	06	
4	22	INV	
5	44	SUM	
6	07	07	
7	69	OP	
8	36	36	
9	69	OP	
30 0	37	37	
1	00	0	
2	42	STO	
3	01	01	
4	02	2	
5	00	0	
6	36	Pgm	
7	09	09	
8	13	C	
9	36	Pgm	
31 0	09	09	
1	14	D	
2	35	1/x	
3	42	STO	
4	08	08	
5	91	R/S	
6			
7			
8			
9			

MERGED CODES

62	72 STO	83 GTO
63	73 RCL	84
64	74 SUM	92 INV SBR

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Fig. 14 Continued

APPENDIX C

Confidence Interval Graph

The program used to generate graphs of the 80% Confidence Intervals of the Beta distribution as a function of the mean and number of trials is contained in this appendix. The data for the limits of the confidence intervals were computed using the International Mathematical and Statistical Library (IMSL) routine MDBETI which evaluates the inverse incomplete Beta function. For the case of 80 percent levels of confidence the incomplete Beta function was evaluated at .1 and at .9 for each of 21 values of the mean for each of the 10 values of N considered. Similar graphs could be generated for other confidence levels by changing the values at which the incomplete Beta function is evaluated.

The exclusion of equal areas from both ends of the distribution will yield the shortest possible confidence interval only for symmetrical distributions. A Beta distribution is symmetrical only when the mean equals .5. It is expected that this graph will be used in the context addressed in the body of the thesis where final means will generally fall in the .4 to .75 region. The skewness of such distributions will not have a large impact on the length of the confidence interval determined from this graph. As an example, a case with 10 samples and a mean of .75 will be subject to a comparatively rather large degree of skewness because of the small sample size and the shift of the mean by .25 from the central value. The shortest confidence interval obtainable for this case does not vary by .01 from the confidence interval obtained by excluding equal areas from both ends of the function. It is not felt that the graph (Figure 15) can be read more accurately than ± .01.

Figure 15 can be used to estimate 80 percent confidence intervals from any Beta distribution for which the mean and the number of trials are known. Use of the graph will be illustrated from one of the cases analyzed in the body of the report. The results of the basic data in the lookdown mode for the micro-model led to a mean of .50 and an N parameter of 15.9. A straight edge can be laid vertically along a line representing the mean (Ⓐ). The intersection of the straight edge with appropriate values of the N curves defines the limits of the confidence interval (Ⓑ and Ⓒ) which are read from the vertical scale on the left of the graph (Ⓓ and Ⓔ). In this case, the straight edge intersect approximate values of N = 15.9 at .34 and .66. A Simpson's rule approximation of areas from 0.0 to .34 and from .66 to 1.0 yielded .0976 and .0988, respectively.

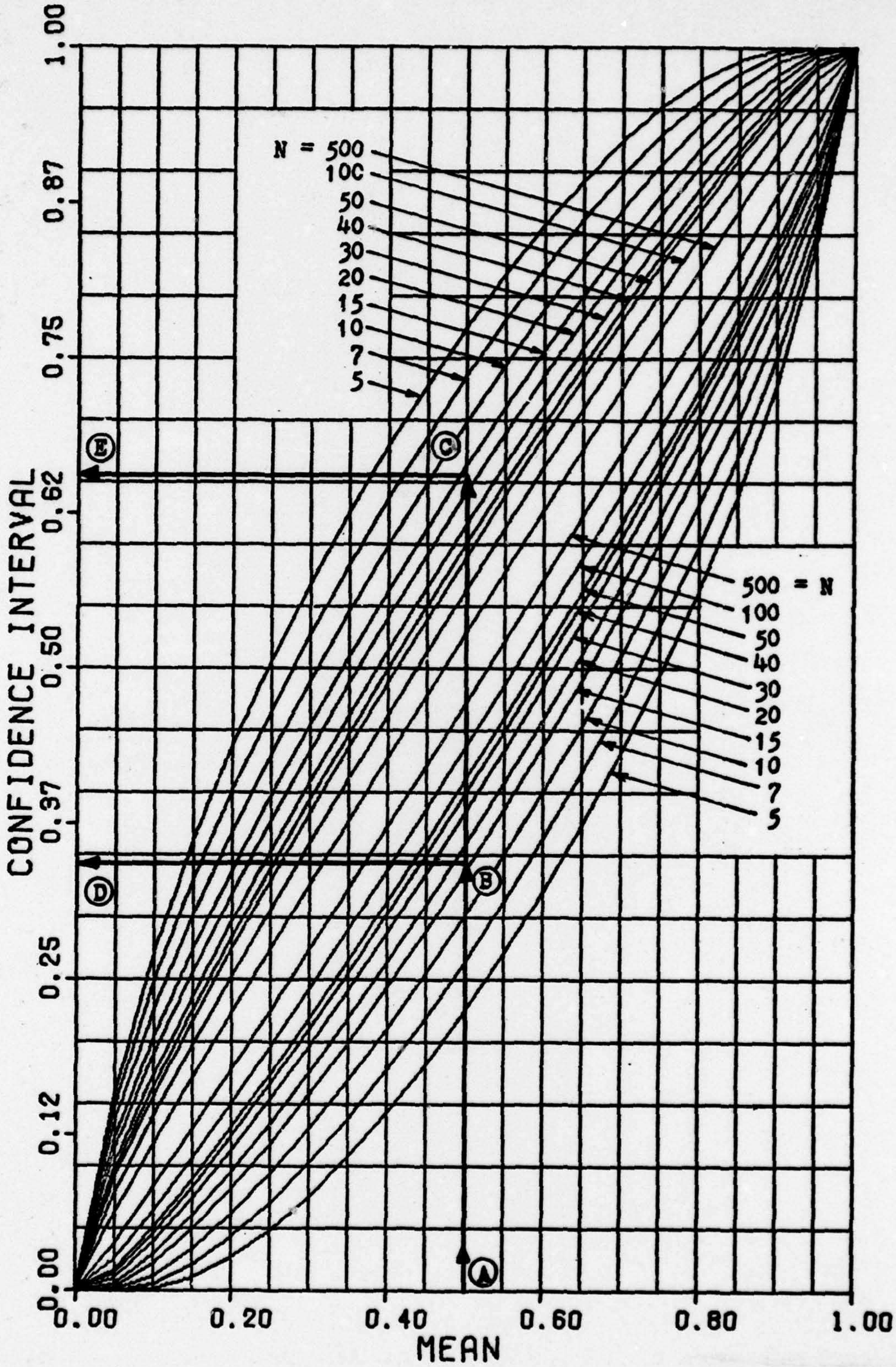

Fig. 15 80% Confidence Intervals for Beta Distribution

```
      PROGRAM CI(INPUT,OUTPUT,TAPE7,TAPE5=INPUT,TAPE6=OUTPUT,
     %PLOT)
      DIMENSION AN(10),Y1(23),Y2(23),X(23)
      DATA AN/5.,7.,10.,15.,20.,30.,40.,50.,100.,500./
100   FORMAT(5F10.5)
110   FORMAT("0",5X,"MEAN",6X,"LWR LMT",3X,"UPR LMT")
120   FORMAT(" N = ",F4.0)
130   FORMAT(F10.3)
140   FORMAT(3F10.5)
      Y1(22)=0.
      Y2(22)=0.
      X(22)=0.
      Y1(23)=.125
      Y2(23)=.125
      X(23)=.2
      CALL PLOTS(30)
      CALL PLOT(0.,-3.,-3)
      CALL PLOT(0.,1.,-3)
      CALL AXIS(0.,0.,4HMEAN,-4,5.,0.,0.,.2)
      CALL AXIS(0.,0.,19HCONFIDENCE INTERVAL,19,8.,
     $90.,0.,.125)
      CALL PLOT(0.,8.,3)
      CALL PLOT(5.,8.,2)
      CALL PLOT(5.,0.,2)
      DO 20 I1=1,10
      ANUM=AN(I1)
      WRITE(6,120)ANUM
      WRITE(6,110)
      Y1(1)=0.
      Y2(1)=0.
      X(1)=0.
      IF(ANUM.EQ.0.)GO TO 20
      DO 10 I2=2,20
      CNT=FLOAT(I2)
      X(I2)=.05*CNT-.05
      AMEAN=X(I2)
      ALPHA=AMEAN*ANUM
      BETA=ANUM-ALPHA
      IF(AMEAN.EQ.0..OR.AMEAN.EQ.1.)GO TO 10
      CALL MDBETI(.1,ALPHA,BETA,Y1(I2),IER)
      CALL MDBETI(.9,ALPHA,BETA,Y2(I2),IER)
10    CONTINUE
      Y1(21)=1.
      Y2(21)=1.
      X(21)=1.
      CALL LINE(X,Y1,21,1,0,72)
      CALL LINE(X,Y2,21,1,0,72)
      DO 15 I3=1,23
      WRITE(6,140)X(I3),Y1(I3),Y2(I3)
15    CONTINUE
20    CONTINUE
```

```
      DO 30 I=1,19
      XDELTA=FLOAT(I)*.25
      CALL PLOT(XDELTA,0.,3)
      CALL PLOT(XDELTA,8.,2)
30    CONTINUE
      DO 40 I=1,19
      YDELTA=FLOAT(I)*.4
      CALL PLOT(0.,YDELTA,3)
      CALL PLOT(5.,YDELTA,2)
40    CONTINUE
      CALL PLOTE(N)
      STOP
      END
```

Vita

Carl J. Vogel was born in St. Louis, Missouri on September 4, 1944. He received his commission upon graduating from the United States Air Force Academy in 1966.

After completing pilot training in Laredo, Texas, he served a tour of duty as a C-7A pilot in the Republic of South Vietnam. He has flown KC-135A's and T-43's. Other duties related to these operational flying assignments have included duty as a Wing Flying Safety Officer, Section Commander, and Chief of a Routes and Airspace Branch. He earned a Master's degree in System's Management from the University of Southern California in 1972.

He is married to the former Lois J. Krater of St. Louis, Missouri. They have four children, Chuck Jr., Christine, Steven, and Michael.

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