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TRAIL: A SHIP-TRAILING MODEL

Maurice M. Mizrahi

CNA

CENTER FOR NAVAL ANALYSES

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This Research Contribution describes a methodology for assessing enemy ability to trail friendly ships at sea. It consists of four parts. The first part treats the search for a lost quarry by shipborne helicopter or long-range reconnaissance aircraft. The second describes a Markov model yielding the fraction of time the ship is free of trail. The third part estimates enemy aircraft requirements to achieve specific search results. The last part presents and documents an APL program, TRAIL, that performs all required calculations.

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Maurice M. Mizrahi

Naval Studies Group

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ABSTRACT

This Research Contribution describes a methodology for assessing enemy ability to trail friendly ships at sea. It consists of four parts. The first part treats the search for a lost quarry by shipborne helicopter or long-range reconnaissance aircraft. The second describes a Markov model yielding the fraction of time the ship is free of trail. The third part estimates enemy aircraft requirements to achieve specific search results. The last part presents and documents an APL program, TRAIL, that performs all required calculations.

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TRAIL: A SHIP-TRAILING MODEL

This Research Contribution describes a methodology to assess enemy ability to trail friendly ships at sea. It consists of four parts. The first part treats aircraft or helicopter search for a lost contact. The second describes a Markov model yielding the fraction of time the ship is free of trail. The third part estimates enemy aircraft requirements to achieve specific search results. The last part presents and documents an APL program, TRAIL, that performs all required calculations.

SEARCH BY AIRBORNE RECONNAISSANCE PLATFORMS

The trailing scenario is as follows. Each friendly ship has an enemy ship assigned to trail it wherever it goes. When trail is broken, a helicopter flies from the trailing ship to search for the target. If the search fails, long-range radar reconnaissance aircraft are summoned from distant bases.

The General Case

Suppose the friendly ship breaks trail at time $t = 0$. We assume here that the air search is random with a uniform target distribution (that is, the target may be found anywhere in its area of uncertainty with equal probability). Let:

$S(t) \equiv$ cumulative area searched up to time t

$A(t) \equiv$ target's area of uncertainty at time t

$P(t) \equiv$ probability of finding target before time t.

Then we have the Koopman formula:

$$P(T) = 1 - K \exp \int_0^T - \frac{S'(t)}{A(t)}\, dt \ , \qquad (1)$$

where K is determined by the initial condition. This formula is proved as follows. Let $Q(t) \equiv 1 - P(t)$. The probability $Q(t + \Delta t)$ of not finding the target between times t and $t + \Delta t$ = [the probability $Q(t)$ of not finding it within time t] x [1 - the probability u of finding it between times t and $t + \Delta t$]. We have $u = [S(t + \Delta t) - S(t)]/A(t)$ because of the uniform target distribution. These results lead to the differential equation $Q'/Q = -S'/A$, whose solution is: Q is proportional to $\exp(-\int S' dt/A)$. The stated result follows.

In particular, let

$T_o \equiv$ time when search begins.

Then S(t) is of the form:

$$S(t) = Y(t - T_o)S_o(t) \quad , \tag{2}$$

where Y(x) is the step function, equal to 1 for x > 0 and equal to 0 otherwise. The initial condition is:

$$P(T_o) = \min[1, S_o(T_o)/A(T_o)] \quad . \tag{3}$$

The derivative of Y(x) is the delta function δ(x), such that $\int_a^b \delta(x - c)f(x)dx \equiv f(c)$ if c is in the interval (a,b) and 0 otherwise. Applying this result to equation 1, we obtain:

$$P(T) = Y(T - T_o)\ P_o(T) \quad , \tag{4}$$

where:

$$P_o(T) \equiv 1 - \left\{1 - \min\left[1, \frac{S_o(T_o)}{A(T_o)}\right]\right\} \exp\left[-\int_{T_o}^{T} \frac{S_o'(t)}{A(t)}\, dt\right] . \tag{5}$$

<u>A Special Case</u>

Let:

v ≡ target speed after it breaks trail

v' ≡ searcher's speed

r ≡ searcher's radar detection range.

Assuming that the target's course is random, the target can be found after time t anywhere inside a circle of radius vt. Thus:

$$A(t) = \pi v^2 t^2 \quad . \tag{6}$$

The searcher sweeps out equal areas in equal times, beginning at T_o, and so:

$$S(t) = 2rv'(t - T_o)Y(t - T_o) \quad . \tag{7}$$

According to equation 5, we have:

$$P_o(T) = 1 - c_1 \exp(c_2/T) \quad , \tag{8}$$

$$\text{with } c_2 \equiv \frac{2rv'}{\pi v^2} \text{ and } c_1 \equiv \exp(-c_2/T_o) \quad . \tag{9}$$

Note that r, v, and v' enter only through the combination rv'/v^2. Thus, variations in r, v, and v' that leave rv'/v^2 unchanged do not affect the search results.

A plot of the detection probability P(T) versus T is found in figure 1. As can be seen, P(T) increases very rapidly with T when the search begins at time T_o. Because the area of uncertainty grows

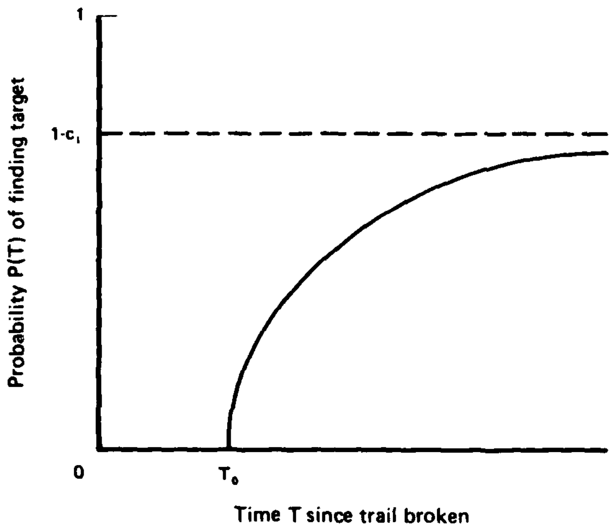


FIG. 1: BEHAVIOR OF AIR SEARCH PROBABILITY DISTRIBUTION P(T)

faster than the searcher's search rate (quadratically versus linearly), the target is not certain to be found. Indeed, letting T tend to ∞ in equation 8 shows that the probability of ever finding the target is:

$$P_{max} = 1 - c_1 \quad , \tag{10}$$

which is smaller than 1.

For helicopter searches immediately following loss of trail, we assume that the ship's area of uncertainty expands at a rate v_2 away from the trailer and at a lower rate v_1 towards the trailer (see figure 2). The difference is due to the fact that the ship is less likely to go in the general direction of the trailer. Assuming the higher noise produced by the accelerating ship does not reduce its area of uncertainty, we see that we must simply replace v^2 with $(v_1^2 + v_2^2)/2$. in equation 9. If the helicopter stops at time T_{max} from trail-break, and there are n_{opp} independent opportunities per day to break trail, and circumstances propitious for trail-breaking occur independently every day with probability w, then the daily probability of breaking trail is:

$$P = w \left[1 - P_o(T_{max})^{n_{opp}} \right] \quad . \tag{11}$$

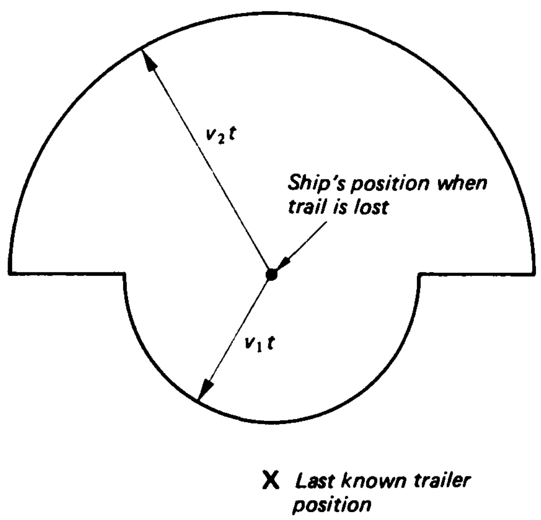


FIG. 2: SHIP'S AREA OF UNCERTAINTY AT TIME t

The probability w can be tied to weather conditions, such as sea state or visibility, or to availability of escorts, or to other factors.

Expected Time Target Is Found

Because the target is not certain to be found, the expected time the target is found is infinite. Nevertheless, one can define a conditional expectation time:

$E(T') \equiv$ expected time target is found given it is found before T'.

It is given by the expected time the target is found up to time T' divided by the probability $P(T')$ of finding it before T', that is:

$$E(T') = \frac{Y(T' - T_o)}{P(T')} \int_0^{T'} TP'(T)dT \ . \tag{12}$$

The result is:

$$E(T') = \frac{c_1 c_2}{1 - c_1 \exp(c_2/T')} \left[\ln \frac{T'}{T_o} + f\left(\frac{c_2}{T_o}\right) e^{c_2/T_o} - f\left(\frac{c_2}{T'}\right) c^{c_2/T'} \right] , \tag{13}$$

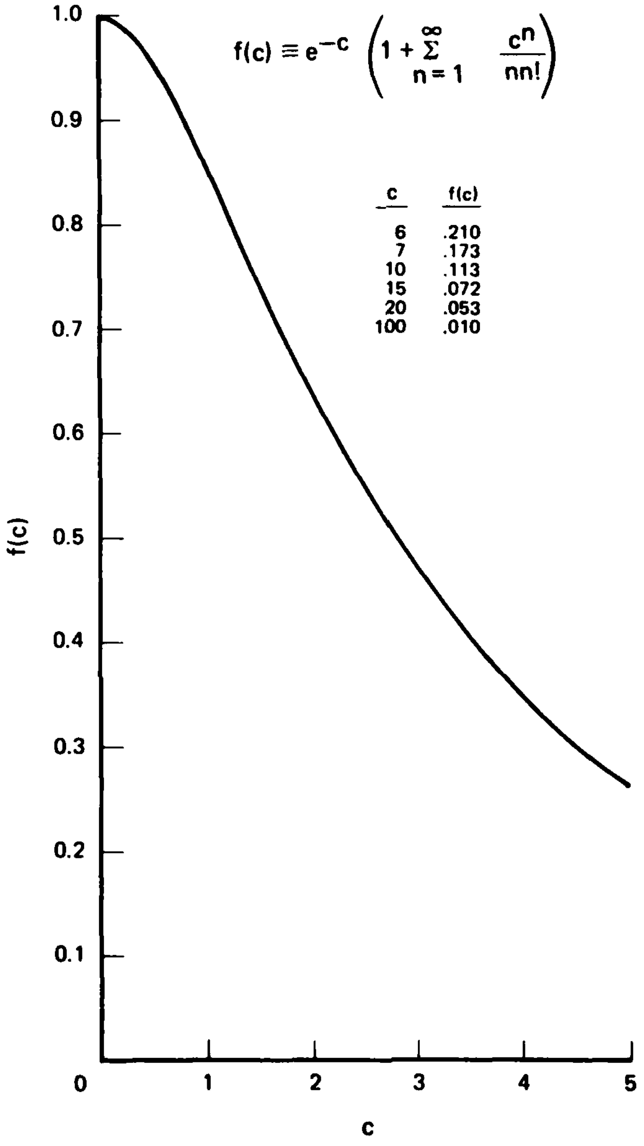
where:

$$f(c) \equiv e^{-c} \left(1 + \sum_{n=1}^{\infty} \frac{c^n}{nn!} \right) . \tag{14}$$

The value of $f(c)$ must be calculated numerically. It is plotted and tabulated in figure 3. All we need is the summation term, but we introduce $f(c)$ because it varies much more gently with c than the summation term. The latter converges rapidly, especially when c is smaller than about 4. A word of caution: the function $f(c)$ is sometimes needed to a high degree of accuracy (several decimal places), so that figure 3 will not always suffice. (Example: when $\exp(c_2/T_o)$ is very large.) A TI-59 calculator program to compute $f(c)$ is provided in table 1. Published tables can also be used (see below).

This result is derived as follows. The derivative P' of P is:

$$P'(T) = \delta(T - T_o)\, P_o(T) + Y(T - T_o)\, P_o'(T) \ , \tag{15}$$



c	f(c)
0.0	1.0000
0.1	.9976
0.2	.9910
0.3	.9809
0.4	.9678
0.5	.9523
0.6	.9349
0.7	.9159
0.8	.8957
0.9	.8745
1.0	.8527
1.1	.8305
1.2	.8080
1.3	.7854
1.4	.7629
1.5	.7405
1.6	.7184
1.7	.6966
1.8	.6752
1.9	.6543
2.0	.6339
2.1	.6140
2.2	.5947
2.3	.5759
2.4	.5577
2.5	.5401
2.6	.5231
2.7	.5067
2.8	.4909
2.9	.4756
3.0	.4609
3.1	.4468
3.2	.4332
3.3	.4201
3.4	.4075
3.5	.3955
3.6	.3839
3.7	.3727
3.8	.3620
3.9	.3518
4.0	.3419
4.1	.3325
4.2	.3234
4.3	.3147
4.4	.3063
4.5	.2983
4.6	.2906
4.7	.2832
4.8	.2761
4.9	.2693
5.0	.2628

FIG. 3: THE FUNCTION f(c)

TABLE 1

TI-59 PROGRAM TO CALCULATE f(c)

Definition: $f(c) \equiv e^{-c} \left(1 + \sum_{n=1}^{\infty} \frac{c^n}{n n!}\right)$

Listing:

```
000  42  STO
001  10   10
002  01   1
003  42  STO
004  06   06
005  42  STO
006  07   07
007  43  RCL
008  10   10
009  94  +/-
010  22  INV
011  23  LNX
012  42  STO
013  08   08
014  76  LBL
015  87  IFF
016  43  RCL
017  10   10
018  94  +/-
019  22  INV
020  23  LNX
021  65   ×
022  53   (
023  43  RCL
024  10   10
025  45  Y^X
026  43  RCL
027  06   06
028  54   )
029  55   ÷
030  43  RCL
031  06   06
032  55   ÷
033  43  RCL
034  07   07
035  95   =
036  85   +
037  43  RCL
038  08   08
039  95   =
040  42  STO
041  08   08
042  66  PAU
043  43  RCL
044  06   06
045  85   +
046  01   1
047  95   =
048  42  STO
049  06   06
050  10  E'
051  61  GTO
052  87  IFF
053  76  LBL
054  10  E'
055  36  PGM
056  16   16
057  11   A
058  36  PGM
059  16   16
060  13   C
061  42  STO
062  07   07
063  92  RTN
```

To use: Key in c, and then press R/S. Display will flash the partial sums for visual inspection of convergence. Stop execution (R/S) when result does not seem to grow bigger. The result is in register 8; the number of terms added in register 6. Program 16 of the master library issued to calculate n!. Registers 1 through 10 and labels A through E, E', and 1' are used. The constant c is in register 10, and n! is in register 7. The larger c is, the longer it takes to get the result.

with P_o in equation 8. This gives (for $T' > T_o$):

$$E(T') = T_o \frac{P_o(T_o)}{P_o(T')} + \frac{1}{P_o(T')} \int_{T_o}^{T'} T P_o'(T)\, dT . \tag{16}$$

This is a general result for all $P(T)$ of the form of equation 4. When $T' \to T_o$, it can be shown that $E(T') \to T_o$, as expected. It can also be shown that $E(T')$ is between T_o and T', as expected. In this special case, the first term is 0 because $P_o(T_o) = 0$ (from equation 8). Expanding the exponential in the integrand term by term gives:

$$\int_{T_o}^{T'} T P_o'(T)\, dT = c_1 c_2 \left(\ln \frac{T'}{T_o} + \sum_{n=1}^{\infty} \frac{c_2^n}{n!} \int_{T_o}^{T'} T^{-n-1}\, dT \right) \tag{17}$$

and the stated result follows. The sum can be related to the exponential-integral function $Ei(x)$ by [1, p. 229, 5.1.10]:

$$\sum_{n=1}^{\infty} \frac{c^n}{n n!} = Ei(c) - \gamma - \ln c , \tag{18}$$

where $\gamma \simeq .5772$ is Catalan's constant and $Ei(c)$ is tabulated [1, pp. 238 ff].

When $P(\infty)$ is close to 1 (that is, when the searcher is practically certain to find its target) it is tempting to let T' tend to ∞ in equation 13 to get the expected time the target is found. Unfortunately, $E(T')$ diverges because of the $\ln(T')$ term. However, it is a mild divergence, so that letting T' be a large but reasonable constant gives the desired expected time, and the latter does not increase much with large increases in T'.

Resumption of trail

Once the aircraft finds the ship, only half the work is done. The trailing ship must now resume trail. What was the trailer doing after trail was broken? Because, in the long run, any course the trailer takes is equally likely to be toward the final target position (at the time the aircraft finds it) as away from it, we assume the trailer roughly maintains its initial position 0 (see figure 4). At the time T the ship is found, the area of uncertainty has grown to a circle (C) of radius vT centered at 0. The ship is found, on average, at a point a distance R_o from 0 such that the circle (C_o) of radius R_o contains half the area of uncertainty. This gives:

$$R_o = vT/\sqrt{2} . \tag{19}$$

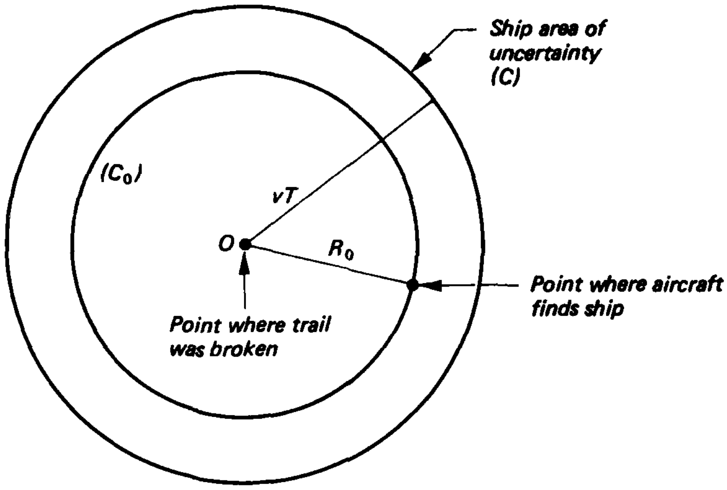


FIG. 4: TYPICAL GEOMETRY WHEN AIRCRAFT FINDS SHIP

Once the aircraft communicates the ship's position to the trailer, the trailer moves toward the ship. But the ship moves too. In the long interval after trail is broken, the ship might receive intelligence information on the trailer's movements, or at least an estimate of the trailer's position when the aircraft detected the ship, and it could move away from that position. If so, it may be a long time before the ship is back under trail, if it ever is. Assume a ship-quarry closure rate of v_T. Then, if the aircraft detects the ship at time T, the ship is back under trail at time kT, where:

$$k \equiv 1 + \frac{vT/\sqrt{2}}{v_T} \quad . \tag{20}$$

More than one aircraft may be needed for any one mission, with each aircraft relaying its findings to its relief. Note that the time between T and kT, when the aircraft is holding the ship and waiting for the trailer to arrive, can in some sense be considered a time under trail for purposes of possible attack by platforms other than the trailing ship. In our context, however, "trail" refers only to direct trail by a specific trailing ship unassisted by outside assets.

Expected Rate of Trail Reacquisition

Our model assumes a constant probability per unit time that a trailer will reacquire its target ship after trail is broken. Thus, if s is the probability of trail being resumed n_1 days after it is lost, then the daily trail-reacquisition probability P' is such that:

$$1 - (1 - P')^{n_1} = s \quad , \tag{21}$$

the days being assumed independent. Solving the equation for P' when s is known, we get:

$$P' = 1 - (1 - s)^{1/n_1} \quad . \tag{22}$$

Here, the ship is first reacquired by an aircraft at time T, distributed according to equation 4. After the aircraft has found the ship, it stays with it until the trailer can arrive and resume trail, which, as described above, takes a period of time proportional to T. Thus, let:

k ≡ constant such that trail is resumed at time kT if aircraft finds ship at time T.

s_o ≡ ratio of desired unit of time for reacquisition rate P' to unit of time in distribution $P(T)$ (here, $s_o = 24$ since we want a rate per day and T is in hours).

$q \equiv s_o/k$.

What we want is the expectation value P' of:

$$1 - [1 - P(T)]^{q/T} \tag{23}$$

relative to the distribution $P(T)$ in equation 4. For the reasons explained earlier, we assumed that the search will be called off at a certain elapsed time τ . This time is designed to ensure that the ship will be found with a given probability p, smaller than $1 - c_1$. τ is therefore given by inverting equation 8:

$$\tau = \frac{c_2}{\ln\left(\frac{1 - p}{c_1}\right)} \quad . \tag{24}$$

We have:

$$P' = \frac{1}{P(\tau)} \int_0^{\tau} \left\{ 1 - \left[1 - P\,(T)\right]^{q/T} \right\} P'(T)\, dT \ . \tag{25}$$

Using equation 15, we have:

$$\begin{aligned} P' = 1 &- \left[1 - P_o(T_o)\right]^{q/T_o} \frac{P_o(T_o)}{P_o(\tau)} \\ &- \frac{1}{P_o(\tau)} \int_{T_o}^{\tau} \left[1 - P_o(T)\right]^{q/T} P_o'\,(T)\, dT \ , \end{aligned} \tag{26}$$

which is valid for all P_o. Specializing to the distribution in equation 8, we get:

$$\begin{aligned} P' = 1 &- c_1^{q/T_o} \exp\left(c_2 q/T_o^2\right) \left[1 - c_1 \exp\left(c_2/T_o\right)\right] P_o^{-1}(\tau) \\ &- P_o^{-1}\,(\tau) \int_{T_o}^{\tau} c_1^{q/T} \left[\exp\left(c_2 q/T^2\right)\right] c_1 c_2 T^{-2} \exp\left(c_2/T\right) dT \ . \end{aligned} \tag{27}$$

Specializing further using equation 9, which indicates that $P_o(T_o) = 0$, we have, with the change of variable $x = 1/T$,

$$P' = 1 - \frac{c_1 c_2}{p} \int_{1/\tau}^{1/T_o} dx \exp\left[\frac{s_o c_2}{k} x^2 + \left(c_2 + \frac{s_o}{k} \ln c_1\right) x\right] . \tag{28}$$

This integral must be calculated numerically (see table 2 for an APL program). Note that, unlike the expected time to find the target we calculated earlier, P' does tend to a limit when $\tau \to \infty$.

FRACTION OF TIME FREE OF TRAIL

The primary measure of effectiveness in avoiding trail is the fraction of time at sea the ship is free of trail.

Result

We now describe the Markov model that leads to the following result:

$E \equiv$ fraction of time at sea the ship is free of trail

$$= \frac{P}{P + P'} + \frac{(P' - P_o P' - P_o P)\,(1 - P - P')\left[1 - (1 - P - P')^n\right]}{n(P + P')^2} \ , \tag{29}$$

TABLE 2

NUMERICAL INTEGRATION[a]

Listing:

```
      ∇INTEGRAL[⎕]∇
     ∇ INT←AB INTEGRAL EPS;H;W;K;L
[1]    W←1⍴0.5×((K←0)++/INTEGRAND AB)×H←-/⌽AB
[2]   F:W←W,(W[(⍴W)-K-1]÷2)+H×+/INTEGRAND(1↑AB)+(H←H÷2)×¯1+2×
       ⍳2⋆¯1+K←K+1+L←0
[3]   G:W←W,INT←(((4⋆L)×W[⍴W])-W[(⍴W)-K])÷¯1+4⋆L←L+1
[4]    →G×⍳L<K
[5]    →F×⍳EPS<|-/W[(⍴W←(-1+2×K)↑W)-0,K]
     ∇
```

How To Run: To calculate $\int_a^b f(x)\,dx$, define the function $f(x)$ in APL (calling it INTEGRAND), input the two-vector AB = (a,b) and the desired precision EPS, and then key in AB INTEGRAL EPS. Vector origin must be 1 (⎕ IO ← 1).

Example: $a = 0$, $b = 1/8.4$, $EPS = 10^{-6}$,
$f(x) = \exp\,[239.5x^2 + (38.2 + 6.27 \ln c_1)\,x]$

```
        ∇INTEGRAND[⎕]∇
       ∇ Y←INTEGRAND X
[1]      Y←⋆(239.5×X⋆2)+X×38.2+6.27×⍟C1
       ∇
        ⎕IO←1
        C1←.0106
        (0,÷8.4) INTEGRAL 1E¯6
1.6409036
```

[a] Using Romberg's rule. Program is from CNA's APL Library.

where:

P_o ≡ probability the ship is trailed when it leaves port

P ≡ probability the ship will shake free on a day when it starts out trailed (assumed constant)

P' ≡ probability the ship is reacquired by its assigned trailer on a day when it starts out free (assumed constant)

n ≡ number of days the trip lasts.

The first term frequently dominates. A required correction to E will be discussed shortly.

Assumptions

The time unit is arbitrary. We chose to use the day. This means that in a given day, the ship is assumed to be either trailed all the time or free all the time. In other words, we assume that the process of breaking trail or reacquiring trail occurs instantaneously at the beginning of a day. This simplifying artifice does not affect the result. If the unit of time is chosen to be, say, the hour, then the "rates"* P and P' must be divided by 24** and the time interval n multiplied by 24. It can be verified that the resulting E remains virtually unchanged.

Note that the daily break-trail probability P refers to losing trail for a large fraction of a day. Broken trails that are quickly reacquired, say by helicopters operating from trailing ships, are not counted. Equation 11 gives a possible estimate of P.

* Strictly speaking, P and P' are transition probabilities, not rates—a subtle but important difference in statistical calculations. If P were a rate, there would be a probability $e^{-P}P^n/n!$ of n transitions per day; in fact, there can be only one transition per day, and its probability is P, not Pe^{-P}. However, since the probability of a transition in n days is $1 - (1 - P)^n$, and since this expression can be rewritten $1 - \exp(-\lambda n)$ if one wishes to consider it a Poisson process, one can identify $-\ln(1 - P)$ with the "rate" λ. For small P, $\lambda \approx P$.

** Or replaced by $\bar{P} = 1 - (1 - P)^{1/24}$, assuming independence from hour to hour.

It is also assumed that each day is independent of the previous day for breaking and reacquiring trail.

Range of Validity of Model

Obviously, the closer the exact probability of reacquiring (or breaking) trail over A days resembles $f(A) = 1 - (1 - p)^A$, where p is a constant, the better the model. Note that the probability of reacquisition is 0 for some time and then surges to high values in a short period of time, unlike the even behavior of f(A)* (see figure 5). Nevertheless, the model is still valid, provided both the total trip duration n and the mean time between trail-breaking events are larger than the time A_0 when the two distributions do not differ significantly in value. If these conditions are not met--say if trail is frequently broken--then the frequent initial periods of no-trail will add up significantly, and the model will underestimate the fraction of time the ship is free of trail.

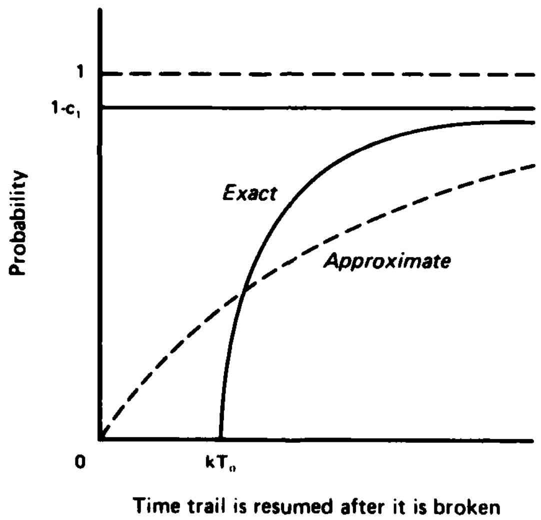


FIG. 5: ILLUSTRATION OF APPROXIMATION IN THE TRAIL MODEL

*The exact probability of trail resumption within A days after trail is broken is 0 if $A<kT_0/s_0$ and $P_0(s_0A/k)$ otherwise, with P_0 as in equation 8.

The Markov Chain

Since the days are assumed independent, we have a Markov process, illustrated in figure 6. It has two states: free (F) and trailed (T). The state vector is $S = (P_F, P_T)$, where P_A is the probability of being in state A. The initial state S_o is:

$$S_o = \left(1 - P_o, P_o\right) \quad . \tag{30}$$

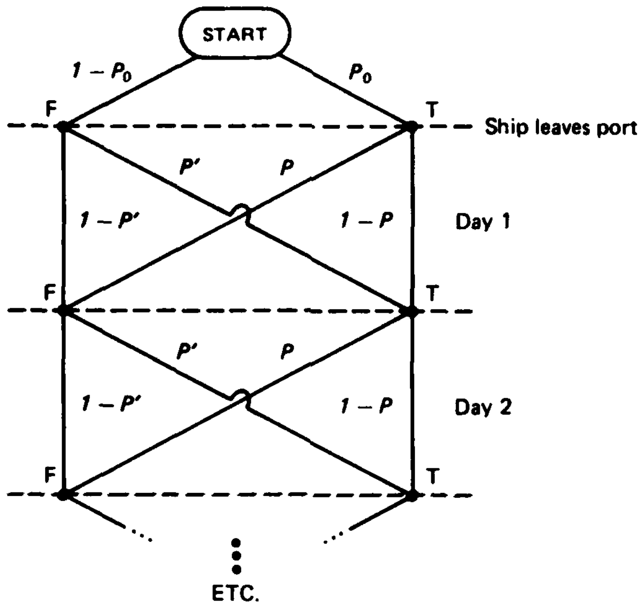


FIG. 6: MARKOV CHAIN FOR TRAILING MODEL

If P_{AB} is the daily probability of transition from state A to state B, then:

$$P_{FF} = 1 - P', \quad P_{FT} = P', \quad P_{TF} = P, \quad P_{TT} = 1 - P \ . \tag{31}$$

The transition matrix M is therefore:

$$M = \begin{pmatrix} 1 - P' & P' \\ P & 1 - P \end{pmatrix} . \tag{32}$$

The state vector S_k after k days is:

$$S_k = S_o M^k \ . \tag{33}$$

The kth power of the matrix M can be calculated* to be:

$$M^k = \frac{1}{P + P'} \begin{pmatrix} P + P'W & P' - P'W \\ P - PW & P' + PW \end{pmatrix}, \tag{34}$$

where:

$$W \equiv (1 - P - P')^k \ . \tag{35}$$

---

*To find the nth power of a 2x2 matrix A, assume $A^n = m_1 A + m_2 I$, where I is the identity matrix. If λ_1 and λ_2 are the eigenvalues and X an eigenvector of A, we have $AX = \lambda X$ and $A^n X = \lambda^n X$, which gives $\lambda^n = m_1 \lambda + m_2$ for $\lambda = \lambda_1$ or $\lambda = \lambda_2$, and yields m_1 and m_2 in terms of λ_1 and λ_2. The result is:

$$m_1 = C_n \text{ and } m_2 = -\lambda_1 \lambda_2 C_{n-1} \ ,$$

which gives:

$$A^n = \begin{pmatrix} aC_n - \lambda_1 \lambda_2 C_{n-1} & bC_n \\ cC_n & dC_n - \lambda_1 \lambda_2 C_{n-1} \end{pmatrix},$$

where:

$$A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C_n \equiv \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \ .$$

The eigenvalues are obtained by writing that the determinant of $A - \lambda I$ is 0. This yields $\lambda_1 = p + q$ and $\lambda_2 = p - q$, where:

$$p = \frac{a + d}{2}, \quad q = \frac{1}{2} \left[(d - a)^2 + 4bc \right]^{1/2} \ .$$

In our case, $\lambda_1 = 1$ and $\lambda_2 = 1 - P - P'$.

The probability Q_k that the ship is free on the kth day is the first element of S_k, namely:

$$Q_k = (P + P')^{-1} \; [P + W(P' - P_oP' - P_oP)] \; . \tag{36}$$

The expected number of days free of trail is simply the sum of the probabilities of being free for each of the days at sea. Dividing by n gives the untrailed fraction E:

$$E = \frac{1}{n} \sum_{k=1}^{n} Q_k \; , \tag{37}$$

which leads directly to the stated result after use of the geometrical progression:

$$\sum_{k=0}^{n} X^k = \frac{1 - X^{n+1}}{1 - X} \; . \tag{38}$$

Special Cases

When $P' = 0$, which is generally the case when the trailer is left to its own devices, the fraction E of time free of trail simplifies to $E = E_o$, where:

$$E_o \equiv 1 - \frac{P_o \; (1 - P)}{nP} \left[1 - (1 - P)^n\right] \; .\text{*} \tag{39}$$

---

* This special case can be simply derived as follows. The probability of shaking the trailer off on the kth day is $P_o(1-P)^{k-1}P$. This action results in $n - k + 1$ days free of trail. Thus:

$$nE_o = n(1 - P_o) + \sum_{k=1}^{n} (n - k + 1)P_o(1 - P)^{k-1}P.$$

Using the geometrical progression result of equation 38 and the fact that

$$\sum_{k=0}^{n} kx^k = x \frac{d}{dx} \sum_{k=0}^{n} x^k = x(1 - x)^{-2} \left[1 - x^n - n(1 - x)x^n\right] ,$$

we get the stated result.

When $n \to \infty$ (long trips), we have:

$$E \to \frac{P}{P + P'} \quad . \tag{40}$$

When $P = P'$ (the ship breaks trail as easily as the trailer reacquires) and $P_o = 1/2$ (the ship is trailed during port exits as often as not), the ship is trailed exactly half the time. The same result obtains if $P = P' = 1/2$, or if $P = P'$ and $n \to \infty$, or if $P = P' = P_o = 1$ and n is even.

Behavior of Q_k

From equation 36 we infer that Q_k, the probability of being free on day k, decreases with k if $P' - P_oP' - P_oP$ is positive and increases with k if that quantity is negative, provided $P + P'$ is smaller than 1. The asymptotic limit (steady-state value) of Q_k for large k is $P/(P + P')$. If $P + P'$ is larger than 1, Q_k oscillates up and down around the limit, but the oscillations damp out with time. If $P + P' = 1$, Q_k is always equal to the limit.

Examples

The behavior of Q_k is illustrated in figure 7 for $P = P'$, $P_o = .2$, and $n = 90$. As can be seen, the closer $P + P'$ is to 1, the faster the process reaches its steady state. The latter is reached rather fast anyway: by the tenth day, Q_k differs from the steady-state value by less than 10 percent.

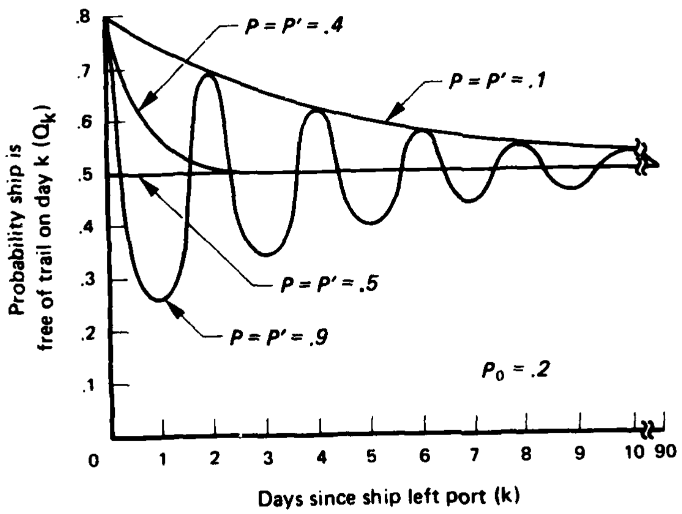


FIG. 7: DAILY PROBABILITY Q_k SHIP IS FREE OF TRAIL

Figure 8 shows how the fraction of time free of trail, E, varies with P, P', and P_o. For simplicity and so as not to introduce bias, we choose P = P'--the ship breaks trail as easily as the trailer reacquires it. The trip lasts n = 90 days. The dependence on P_o, predictable in nature but surprising in magnitude, is as follows:*

- If P is relatively large (over 5 percent a day)--frequent trail loss and reacquisition--the result is virtually independent of P_o. For example, for P = 5 percent, if the ship is picked up in port (P_o = 1), it is free of trail 45 percent of the time; if the ship is not picked up in port, it is free of trail 55 percent of the time.

- If P is relatively small (less than 5 percent a day)--rare trail loss and reacquisition--the result depends strongly on P_o. For example, for P = 1 percent, if the ship is picked up in port (P_o = 1), it is free of trail 27 percent of the time; if the ship is not picked up in port, it is free of trail 73 percent of the time.

Corrected Fraction of Time Free of Trail

If the aircraft search succeeds only with probability p, then equation 28 gives the corresponding P' and equation 29 the corresponding E. Since the ship is not reacquired with probability 1, and since only 98 percent of the trail-breaking events are prosecuted, the (corrected) expected fraction of time free of trail, $\overline{E}$, is:

$$\overline{E} = gp \times E \text{ (calculated with the } P' \text{ corresponding to } p\text{, using equation 28)} + (1 - gp) \times E \text{ (calculated with } P' = 0\text{, from equation 39)} \quad (40a)$$

where g = .98.

* The curves are symmetric with respect to the line $E = 1/2$. That is, if P = P', $E - 1/2$ changes sign when P_o changes into $1 - P_o$.

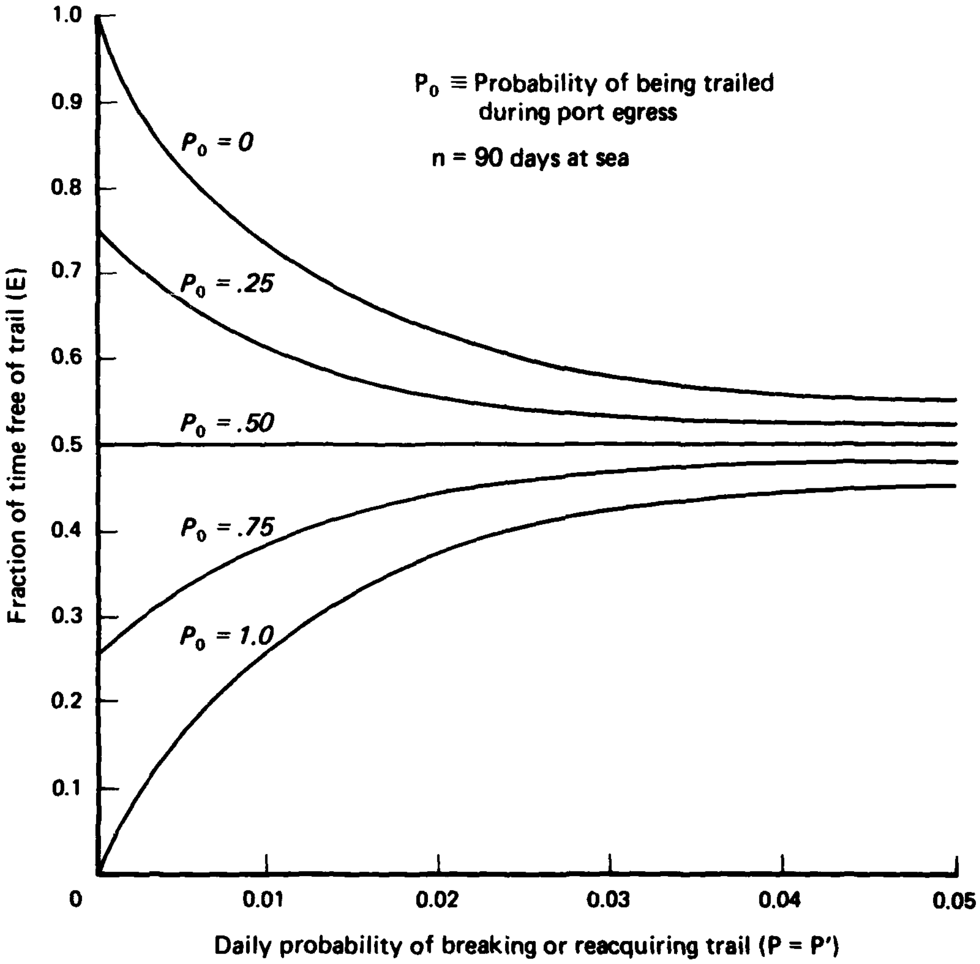


FIG. 8: EFFECT OF TRAILING AT PORT EGRESS

Equation 40a, which is an approximation, is a correction to E required by the observation that a constant daily reacquisition probability P', however small, implies eventual reacquisition, whereas in reality reacquisition is not certain: its probability can be no higher than gp. Equation 40a is a slight underestimate of the fraction of time free of trail.

ENEMY AIRCRAFT REQUIREMENTS

In this section, the number of reconnaissance aircraft the enemy requires to achieve specific search results is calculated.

To estimate the required aircraft inventory for the reacquisition efforts, the quantity

$f_m(k) \equiv$ probability of m trail-breaking events in the first k days of the n-day trip

is needed. It is not easy to calculate.* Here we only present an exact line of approach and an approximation, and calculate the mean of the distribution.

Exact Approach

This method, developed by James K. Tyson, consists of defining the matrix:

$$M_z \equiv \begin{pmatrix} 1 - P' & P' \\ zP & 1 - P \end{pmatrix}, \quad (41)$$

which reduces to the Markov matrix M in equation 32 for $z = 1$. By "flagging" the P with a z, we isolate a trail-breaking event. The coefficient of z^m in any result based on M_z is then associated with m trail-breaking events. Let $S_k \equiv (F_k\ T_k) = S_o M_z^k$ be the state vector after k days. Then $f_m(k)$ is simply the coefficient of z^m in

* $f_m(k)$ is not, as one might think, the binomial distribution $\binom{k}{m} P^m(1 - P)^{k-m}$. It depends on P_o and P' as well. As an illustration, if $P_o = P' = 0$, the ship is never trailed, so it never breaks trail, and $f_m(k) = 0$ for all nonzero m and k.

$F_k + T_k$ (we sum the two because it does not matter whether the end state is free or trailed), that is:

$$f_m(k) = \frac{1}{m!}\left[\frac{\partial^m}{\partial z^m}\,(F_k + T_k)\right]_{z=0}. \tag{42}$$

The eigenvalues of M_z are:

$$\lambda_1 = v + \sqrt{u} \text{ and } \lambda_2 = v - \sqrt{u} \ , \tag{43}$$

where:

$$v \equiv 1 - \frac{P + P'}{2} \ , \quad u \equiv \left(\frac{P - P'}{2}\right)^2 + zPP' \ . \tag{44}$$

We have:

$$(M_z)^k = C_k M_z - \lambda_1 \lambda_2 C_{k-1} I \ , \tag{45}$$

where:

$$C_k \equiv \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} = \sum_{S=1}^{k}{}_{o} \binom{k}{S} v^{k-S} u^{\frac{S-1}{2}} \ , \tag{46}$$

where Σ_o indicates that the summation extends only to odd S. Note that $f_m(k) = 0$ for $m>k/2$. The formula checks out for small values of k, but no simple closed form could be found.

Expected Number of Trail-Breaking Events

The mean of the unknown distribution $f_m(k)$ can be calculated exactly using the results from the Markov approach. We have:

$\bar{f}(k) \equiv$ expected number of trail-breaking events in the first k days

$$= P\,[n(1 - E) + P_o - (1 - Q_k)] \ , \tag{47}$$

with E as in equation 29 and Q_k as in equation 36. The first term generally dominates, especially for large n and small E.

This result is derived as follows. The probability of a trail-breaking event on day 1 is $P_o P$; on day 2 it is $(1 - Q_1)P$; and on day k it is $(1 - Q_{k-1})P$. (The probability of being trailed on day i is $1 - Q_i$). Adding up these probabilities gives $\bar{f}(k)$. Using the definition of E in equation 37, we get the stated result.

Approximation

We base this approximation on equilibrium (steady state) being reached rapidly. According to equation 36, the probability $1 - Q_k$ of being trailed on day k rapidly approaches its steady-state value,

$$1 - Q_k \rightarrow \frac{P'}{P + P'} , \qquad (48)$$

when <u>any</u> of the following three conditions is met:

- $P_o \approx P'/(P + P')$ (the trip happens to begin in the steady state) (49)
- $P + P \approx 1$ (the system falls in the steady state, $Q_k \approx P$, almost from day 1) (50)
- $(1 - P - P')^k \approx 0$ for $k \ll n$ (fast approach to the steady state). (51)

If one of these conditions obtains, we can assume that the probability s of breaking trail on any given day in the steady state,

$$s \approx \frac{P'P}{P + P'} , \qquad (52)$$

is independent from day to day. The probability $f_m(n)$ of m trail-breaking events in the n days is therefore approximated by a binomial distribution:

$$f_m(n) \approx \binom{n}{m} s^m (1 - s)^{n-m} . \qquad (53)$$

Now compare the mean of the above distribution, ns, with the exact mean $\bar{f}(n)$ calculated earlier (equation 47). If the two are in agreement, the approximation is probably good. Generally, agreement will be observed if both these two conditions are met:

- $E \simeq P/(P + P')$ (54)

- $n >> P_o\ (1 + P/P') - 1\ .$ (55)

The first condition obtains for large n or if either of the conditions in equations 49 and 50 is met. The second expresses the fact that the first term should dominate in the expression of $\bar{F}(n)$ in equation 47.

Note that the steady-state distribution of trail reacquisition events is the same as that for trail-breaking events, as it should be. Indeed, $PP'/(P + P')$ can be interpreted as either $Px[P'/(P + P')]$ or $P'x[P/(P + P')]$.

An APL program for calculating the cumulative binomial is given in table 3.

Calculation of Aircraft Requirements

Given the validity of this approximation, the standard deviation σ is:

$$\sigma = [ns(1 - s)]^{1/2} \tag{56}$$

The cumulative distribution, or probability that k or fewer events will occur, is approximated [2] by:

$$\phi\left(\frac{k + 1/2 - ns}{\sigma}\right), \tag{57}$$

where:

$$\phi(x) \equiv \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-t^2/2)dt \tag{58}$$

is the standardized normal distribution. Thus, the probability of $N\sigma$ events over the mean ns is given by:

$$\phi\left(N + \frac{1}{2\sigma}\right)\ . \tag{59}$$

For 2σ confidence ($N = 2$), the probability is over 98 percent, since $\phi(2) = 97.7$. Thus, by planning for 2σ events over the mean, one is confident of being able to handle at least 97.7 percent of the events.

TABLE 3

CUMULATIVE BINOMIAL APL PROGRAM

Coding:

```
     ∇BIN[⎕]∇
    ∇ BIN
[1]    S←(P×PP)÷P+PP
[2]    W←V←1+M←¯1
[3]    'P = ';P;' , PP = ';PP;' , SO S = ';S
[4]    'N = ';N
[5]   A:W←W,V←V+(M!N)×(S*M)×(1-S)*N-M←M+1
[6]    →(M<N)⍴A
[7]    'PROBABILITY OF K EVENTS OR LESS (K = 0,1,2,...,N): ';1↓W
[8]    'MEAN NUMBER OF EVENTS: ';N×S
[9]    'STANDARD DEVIATION: ';(N×S×1-S)*0.5
    ∇
```

Input: N (number of days n), P (probability P), PP (probability P')

Intermediate output: the binomial probability S ≡ PP'/(P + P')

To execute: Key in inputs, then BIN.

Example:

```
P←.243   )
PP←.28   }  key in
N←90     }
BIN      )

P = 0.243 , PP = 0.28 , SO S = 0.1300956
N = 90
PROBABILITY OF K EVENTS OR LESS (K = 0,1,2,...,N):
     3.568103E¯6  0.0000515935  0.0003712048  0.001773291
     0.006333921  0.01806518  0.04291957  0.08752375
     0.1567315  0.2510327  0.3652662  0.489512  0.6118379
     0.7216022  0.811887  0.8802984  0.9282563  0.9594764
     0.9784119  0.989143  0.9948403  0.9976804  0.9990125
     0.9996015  0.9998474  0.9999445  0.9999808  0.9999937
     0.999998  0.9999994  0.9999998  1  1  1  1  1  1  1  1
     1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
     1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
     1  1  1  1  1  1  1  1  1  1  1  1  1  1
MEAN NUMBER OF EVENTS: 11.7086
STANDARD DEVIATION: 3.191452
```

The number of trail-breaking events per n-day trip that are observed 98 percent of the time is therefore:

$$\bar{f}(n) + 2\sqrt{ns(1 - s)} \tag{60}$$

for a single ship. For n' independent ships, the mean is multiplied by n' and the standard deviation by $\sqrt{n'}$, * so that the number is:

$$\bar{\nu} = n'\bar{f}(n) + 2\sqrt{nn's(1 - s)} \quad . \tag{61}$$

It is always best to use aircraft assets to prosecute many trail-breaking events with few sorties per event rather than few trail-breaking events with many sorties per event, because the area of uncertainty grows faster than the area searched.

A conservative estimate (to be discussed) of the number N_R of reconnaissance aircraft required in inventory to guarantee (1) that over 98 percent of events will be prosecuted and (2) that trail will be resumed after each event with probability p, is:

$$N_R = \nu t_1/n_s t_s \quad , \tag{62}$$

where:

n_s = number of sorties an aircraft makes per month

t_s = maximum time an aircraft can stay on station

t_1 = aircraft time on station that guarantees trail is resumed with probability p (= $k\tau - T_o$)

$$\nu = \frac{30}{n}\bar{\nu}, \text{ with } \bar{\nu} \text{ in equation 61} \tag{63}$$

= number of trail-breaking events per month aircraft inventory is tailored to prosecute.

---

* When n independent random variables are summed, adding the means and the variances gives the mean and variance of the sum.

This number is derived as follows. Since t_1 aircraft-hours are required per event and our aircraft can only spend t_s hours on station, $n_A = t_1/t_s$ sorties are needed per event, that is, νn_A sorties per month. Since each aircraft can make only n_s sorties per month, $\nu n_A/n_s$ aircraft are needed.

If each reconnaissance aircraft requires q tankers, the total number of aircraft required is:

$$N_{RT} = (1 + q)N_R \ . \tag{64}$$

The result in equation 62 for enemy aircraft requirements is only a rough, conservative estimate. Many more assumptions, models, tradeoff analyses, and operational data are required for an accurate assessment of those requirements. First, the trail-breaking events are assumed evenly spread in time, even though they may tend to cluster. For example, if trail tends to be broken in bad weather, then bad-weather periods will contain many more events than good-weather periods. These surge requirements will drive up enemy requirements. Consider the example (arbitrary parameters) where five sorties are needed per event for a $p = .6$ reacquisition probability and the enemy plans for $\nu = 85$ trail-breaking events per month under conditions yielding a $w = .12$ daily probability that circumstances will be propitious for trail-breaking. The aircraft fly 12 sorties per month. Our results show that the enemy needs 85 x 5/12 = 35 aircraft. But if we assume perfect correlation between the times when circumstances propitious for trail-breaking occur throughout the ships' area of deployment, trail-breaking events will cluster every 1/.12 = 8.3 days. Thus, as many as 85 x 8.3/30 = 24 events could be expected to occur roughly at the same time, which would drive enemy requirements up to 24 x 5 = 120 aircraft—over three times the number we calculated. Therefore, the detailed distribution of trail-breaking events in time and information on the correlation between the occurrence of circumstances propitious for trail-breaking in different areas are needed. Second, we assumed the enemy knows when to expect reacquisition, when in reality he does not and must automatically send follow-up sorties to ensure that an aircraft about to leave station is relieved. On the other hand, we used the maximum (not average) number of sorties consistent with a given probability of reacquisition (p), because there is a limit to the extent to which unused sorties can be "saved" for future use.*

Information on how frequently an aircraft can be flown in a given time interval, however short, must be included in more refined esimates of enemy requirements. Also, variations that might be attractive for the enemy—such as flying more than one sortie soon after trail is broken, while the area of uncertainty is still small—must be assessed.

As can be expected the dominant factor by far in anticipating enemy aircraft requirements is w — the fraction of time opportunities for trail-breaking occur.

APL PROGRAM FOR TRAIL

Description

Most of the above calculations were encoded in an APL program. The 26 input variables are listed in table 4 and the 23 output variables in table 5. The coding is in table 6. There are four different functions:

- ARM, the overall control function (for aircraft reconnaissance mission)
- INTEGRAL, the function that performs the integral in equation 28
- INTEGRAND, the integrand of equation 28
- PEE, the function that calculates P from equation 11.

The program is executed by keying in all input variables and then ARM. Should the input probability of finding the ship, p, be larger than can possibly occur, an "invalid" message is printed. Vector origin is 1, and results are printed to four significant figures, as instructed in line 2 of ARM. Should the input tolerance EPS used to calculate the

* The following simple example shows that there are cases when only the maximum number of aircraft will satisfy the requirement. Assume there is exactly one trail-breaking event per day, that the aircraft can each fly exactly one sortie per day, and that the ship is reacquired the same day it breaks trail, if at all. Assume further that one sortie yields a .4 probability of reacquisition, two sorties .6, and three sorties .8. The average number of sorties flown is then (.4 x 1) + (.2 x 2) + (.4 x 3) = 2. The question is: how many aircraft in inventory will guarantee that 80 percent of the trail-breaking events are reacquired over a long period of time? The answer is three (the maximum), not two (the average). If only two aircraft are bought, there will be no way a third can be found when needed. Unused sorties are lost forever in this case.

TABLE 4

INPUT VARIABLES IN TRAIL

Name of variable	Symbol for variable in text	Definition
DR	r	Reconnaissance aircraft radar detection range
DRH	r	Trailerborne helicopter radar detection range
EPS	-	Tolerance in calculation of integral in equation 28 (example: .001 for three-decimal-place accuracy)
FRE	-	Frequency vector (weights attached to each of the aircraft mission radii in R, according to the likelihood of each mission radius. Sums to 1.)
HTR	T_o	Trailerborne helicopter reaction time--from breaking of trail to initiation of search
N	n	Number of days trip lasts
NS	N_s	Number of sorties a reconnaissance aircraft makes every month
NP	n'	Number of ships (to be trailed) at sea at any given time
OPP	n_{opp}	Number of opportunities to break trail on a day when circumstances (e.g., weather) are propitious for trail-breaking
PØ	P_o	Probability ship is trailed when exiting port
PS	p	Probability of finding ship by reconnaissance aircraft, on which aircraft inventories are based
R	-	Vector bearing the mission radii the aircraft might be called on to cover

TABLE 4 (Cont'd)

Name of variable	Symbol for variable in text	Definition
SØ	s_o	Number of hours in a day (=24)
S1	s_1	Number of days in a month (=30)
SMX	T_{max}	Time from trail-breaking when trailer-borne helicopter stops searching and calls reconnaissance aircraft
TKF	q	Number of tankers required per reconnaissance aircraft
TR	-	Time from trail-breaking to reconnaissance aircraft taking off (TR>SMX)
TS	t_s	Reconnaissance aircraft time on station for each of the mission radii in R (vector)
V	v	Ship's cruise speed (long-term rate of expansion of ship's radius of uncertainty)
V1	v_1	Rate of expansion of ship's radius of uncertainty in directions towards the trailer, shortly after breaking trail
V2	v_2	Rate of expansion of ship's radius of uncertainty in directions away from the trailer, shortly after breaking trail (usually, ship's maximum sustained speed)
VA	-	Reconnaissance aircraft transit speed to station
VH	v'	Trailerborne helicopter search speed
VP	v'	Reconnaissance aircraft search speed
VT	v_T	Trailer-to-ship closing speed after aircraft has detected ship
WEA	w	Probability that circumstances (e.g. weather) will be propitious for trail-breaking on a given day.

TABLE 5

OUTPUT VARIABLES IN TRAIL

Name of variable	Symbol for variable in text	Definition
C1	c_1	Quantity defined in equation 9 (vector)
C2	c_2	Quantity defined in equation 9
E	E	Uncorrected fraction of time ship is free of trail (equation 29) (vector)
EØ	E_o	Fraction of time ship is free of trail when P' = 0 (no reacquisition capability) (equation 39)
EB	$\bar{E}$	Corrected fraction of time ship is free of trail (equation 40a) (vector)
EBØ	-	Average value of EB
FBN	$\bar{f}(n)$	Expected number of trail-breaking events during ship's n-day trip (equation 47) (vector)
II	-	Value of integral in equation 28 (vector)
KK	k	Constant such that trail resumes at time kT if aircraft detects ship at time T (equation 20)
NR	N_R	Required number of reconnaissance aircraft (equation 62) (vector)
NRT	N_{RT}	Total number of aircraft required, including tankers (equation 64) (vector)
NRTØ	-	Average value of NRT
NU	ν	Number of trail-breaking events per month that is not exceeded more than 2 percent of the time (equation 63) (vector)

TABLE 5 (Cont'd)

UNCLASSIFIED

Name of variable	Symbol for variable in text	Definition
P	p	Effective probability ship shakes free on a day when it starts out trailed (equation 11) (vector)
PH	-	Probability trailerborne helicopter reacquires ship per trail-breaking attempt ($P_o(T_{max})$ in equation 11)
PMAX	P_{max}	Maximum probability of aircraft detecting ship (given unlimited search) (equation 10) (vector)
PP	P'	Effective probability ship is acquired by its trailer on a day when it starts out free (equation 28) (vector)
QN	Q_n	Probability ship is free on day n--the last day of the trip (equation 36) (vector)
S	s	Steady-state daily probability of either breaking or reacquiring trail (equation 52) (vector)
TØ	T_o	Time since trail-breaking when reconnaissance aircraft arrives on station (vector)
T1	t_1	Aircraft time on station per trail-breaking event that guarantees a probability PS (p) of finding ship (=(KK x TAU) - TØ) (vector)
TAU	τ	Aircraft search time that results in a probability PS of finding the ship (equation 24) (vector)
TT	t_t	Reconnaissance aircraft transit time to station (vector)
AB,C11,E1,F,G, H,I,INT,K,L,Q, W,X,X1,X2,Y		Variables used internally

TABLE 6

APL CODING OF TRAIL

```
      ∇ARM[⎕]∇
    ∇ ARM;Q
[1]    'INPUT:'
[2]    ⎕PP←3+⎕IO←1,0⍴G←0.98
[3]    'VA=';VA;', TR=';TR;', VP=';VP;', DR=';DR;', V=';V;',
       PS=';PS;', VT=';VT;', S0=';S0;', EPS=';EPS;',  P0=';P0
       ;', N=';N;',  NP=';NP;', NS=';NS;', S1 = ';S1;', V1=';
       V1;', V2=';V2;', VH=';VH;', DRH=';DRH;', HTR=';HTR;',
       OPP=';OPP;', SMX=';SMX;', WEA=';WEA;' ,TKF = ';TKF
[4]    'R = ';R
[5]    'FRE = ';FRE
[6]    'TS = ';TS;1 6⍴'-'
[7]    'OUTPUT';1 6⍴'-'
[8]    'C2 = ';C2←(2×DR×VP)÷(V*2)×○1
[9]    'KK = ';KK←1+V÷VT×2*0.5
[10]   'AVG TT = ';+/FRE×TT;',TT = ';TT←R÷VA
[11]   'AVG T0 = ';+/FRE×T0;', T0 = ';T0←TT+TR
[12]   'C1 = ';C1←*(0-C2)÷T0
[13]   'AVG PMAX = ';+/FRE×PMAX;', PMAX = ';PMAX←1-C1
[14]   →(PS>⌊/PMAX)⍴X2
[15]   'AVG TAU = ';+/FRE×TAU;', TAU = ';TAU←C2÷⍟(1-PS)÷C1
[16]   I←II←0
[17] X1:AB←(÷TAU[I]),÷T0[I←I+1]
[18]   C11←C1[I]
[19]   II←II,AB INTEGRAL EPS
[20]   →(I<⍴R)⍴X1
[21]   'AVG II = ';+/FRE×II;', II = ';II←1↓II
[22]   'AVG PP = ';+/FRE×PP;', PP = ';PP←1-C1×C2×II÷PS
[23]   PEE
[24]   E1←P+PP
[25]   'AVG E = ';+/FRE×E;', E = ';E←(P÷E1)+((PP-P0×E1)×(1-E1
       )×1-(1-E1)*N)÷N×E1*2
[26]   'E0 = ';E0←1-P0×(1-P)×(1-(1-P)*N)÷N×P
[27]   'AVG EB = ';EB0←+/FRE×EB;', EB = ';EB←((PS×E)+E0×1-PS)
       ÷G
[28]   'AVG QN = ';+/FRE×QN;', QN = ';QN←(P+(PP-P0×E1)×(1-E1)
       *N)÷E1
[29]   'AVG FBN = ';+/FRE×FBN;', FBN = ';FBN←P×(N×1-E)+P0-1-
       QN
```

TABLE 6 (Cont'd)

```
[30]  'AVG S = ';+/FRE×S;', S = ';S←P×PP÷E1
[31]  'AVG NU = ';+/FRE×NU;', NU = ';NU←(S1+N)×(NP×FBN)+2×(|
      NP×N×S×1-S)*0.5
[32]  'AVG T1 = ';+/FRE×T1;', T1 = ';T1←(KK×TAU)-T0
[33]  'AVG NR = ';+/FRE×NR;', NR = ';NR←NU×T1÷NS×TS
[34]  'AVG NRT = ';NRT0←+/FRE×NRT;', NRT = ';NRT←(1+TKF)×NR;
      1 1⍴''
[35]  'SUMMARY; AVG NRT = ';NRT0;', AVG EB = ';EB0;', E0 = '
      ;E0;1 60⍴'-'
[36]  →0
[37] X2:'CASE INVALID, PS IS UNATTAINABLE.';1 60⍴'-'
    ∇
     ∇INTEGRAL[⎕]∇
    ∇ INT←AB INTEGRAL EPS;H;W;K;L
[1]    W←1⍴0.5×((K←0)++/INTEGRAND AB)×H←-/⌽AB
[2]   F:W←W,(W[(⍴W)-K-1]÷2)+H×+/INTEGRAND(1↑AB)+(H←H÷2)×¯1+2×
       ⍳2*¯1+K←K+1+L←0
[3]   G:W←W,INT←(((4*L)×W[⍴W])-W[(⍴W)-K])÷¯1+4*L←L+1
[4]    →G×⍳L<K
[5]    →F×⍳EPS<|-/W[(⍴W←(-1+2×K)↑W)-0,K]
    ∇
     ∇INTEGRAND[⎕]∇
    ∇ Y←INTEGRAND X
[1]    Y←*((S0×C2÷KK)×X*2)+X×C2+(S0÷KK)×⍟C11
    ∇
     ∇PEE[⎕]∇
    ∇ PEE;Q
[1]    Q←(4×DRH×VH)÷○1×(V1*2)+V2*2
[2]    'PH = ';PH←1-*Q×(÷SMX)-÷HTR
[3]    'P = ';P←WEA×1-PH*OPP
    ∇
```

integral be too small, space limits may be reached, and EPS may have to be reduced (which sometimes happens for small ship speeds v because of the large value of the integral).

Several aircraft mission radii can be input at the same time, with their relative frequencies of occurrence. The results are then presented separately per mission radius and averaged.

A word on units. The input and output lists refer to hours, days, and months. This choice is arbitrary and can be changed. The issue here is not preference but reasonableness of results. A ship is in the "trail" or "no-trail" state for a whole day. If it is felt, for example, that a 12-hour interval would be more appropriate than a day; the input SØ must be changed from 24 to 12. The same goes for the month, which is the time interval on which aircraft inventories are based (input S1).

<u>Examples</u>

A sample run of TRAIL is shown in table 7. The input quantities are arbitrary and are not intended to reflect specific systems.

An example of the kind of results TRAIL can generate is the fraction of ships at sea that are free of trail at any given time, $\bar{E}$, as a function of the number of aircraft the enemy deploys. This function is obtained by running TRAIL several times for different values of p (the trail reacquisition probability that the enemy tailors his aircraft inventory to achieve). In our illustrative case, the result is:

p	(PS)	0	.1	.3	.5	.7	.8	.9	.95	.99
n_{RT}	(NRT)	0	12	18	22	26	29	36	45	106
$\bar{E}$	(EB)	.84	.82	.68	.52	.35	.27	.18	.14	.11

Figure 9 is a plot of $\bar{E}$ versus n_{RT}. When no aircraft are deployed, the trailing ships are left to their own devices.

TABLE 7

SAMPLE RUN OF TRAIL

```
     ARM
INPUT:
VA=650, TF=5, VP=150, DR=250, V=20, PS=0.7, VT=3, S0=24, EPS
     =1E-5,  P0=0.5, N=120,  NP=50, NS=10, S1 = 30, V1=10, V
     2=30, VH=75, DRH=40, HTR=1.2, OPP=4, SMX=3, WEA=0.05 ,T
     KF = 0.9
R = 2000  3000  4000
FRE = 0.2  0.7  0.1
TS = 25  15  10
-------
OUTPUT
-------
P2 = 59.68
KF = 5.714
AVG TT = 4.462,TT = 3.077  4.615  6.154
AVG TO = 9.462, TO = 8.077  9.615  11.15
P1 = 0.0006178  0.002015  0.004744
AVG PMAX = 0.998, PMAX = 0.9994  0.998  0.9953
AVG TAU = 11.72, TAU = 9.649  11.93  14.39
AVG IT = 6.674, II = 15.29  4.862  2.122
AVG PP = 0.1683, PF = 0.1946  0.1646  0.1417
PH = 0.8519
F = 0.02367
AVG E = 0.1372, E = 0.1201  0.1392  0.1581
E0 = 0.8378
AVG EB = 0.3545, EB = 0.3423  0.3559  0.3694
AVG QN = 0.124, QN = 0.1084  0.1257  0.1431
AVG FBN = 2.441, FBN = 2.49  2.436  2.383
AVG S = 0.02073, S = 0.0211  0.02069  0.02028
AVG NU = 36.03, NU = 36.69  35.96  35.24
AVG T1 = 57.5, T1 = 47.06  58.55  71.08
AVG NR = 13.71, NR = 6.905  14.04  25.05
AVG NRT = 26.05, NRT = 13.12  26.67  47.6

SUMMARY: AVG NRT = 26.05, AVG EB = 0.3545, E0 = 0.8378
-----------------------------------------------------------
```

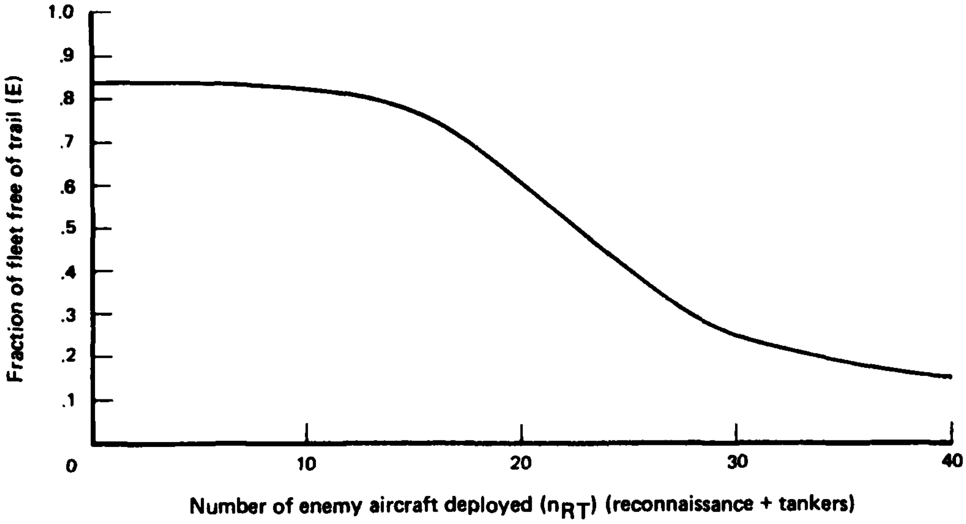


FIG. 9: SAMPLE RESULTS OBTAINED FROM TRAIL RUNS[a]

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[a] *Based on a fleet of 50 target ships and 50 trailing ships at sea all the time.*

REFERENCES

[1] Abramowitz, M., and Stegun, I.A. Handbook of Mathematical Functions. New York: Dover Publications, 1968

[2] Feller, William. An Introduction to Probability Theory and Its Applications. Vol. I., 3rd edition. New York: Wiley, 1968