NAVAL POSTGRADUATE SCHOOL
Monterey, California

DTIC
ELECTE
MAY 1 8 1983

A

PROGRAMS FOR A TARGET

POSITION ESTIMATION PROCEDURE

BY

R. N. FORREST

March 1983

Approved for public release; distribution unlimited.

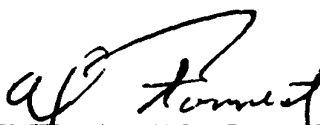Prepared for: Naval Postgraduate School
Monterey, CA 93940

83   05 18 046

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral J. J. Ekelund   David A. Schrady
Superintendent   Provost

Reproduction of all or part of this report is authorized.

This report prepared by:

R. N. FORREST, Chairman
Antisubmarine Warfare
Academic Group

Reviewed by: Released by:

D. A. SCHRADY   W. M. TOLLES
Provost   Dean of Research

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS71-83-002	2. GOVT ACCESSION NO. AD-A128 256	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Programs for a Target Position Estimation Procedure		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. N. Forrest		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, Ca 93940		12. REPORT DATE March 1983
		13. NUMBER OF PAGES 48
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Bearings-only Position Estimation
Calculator Programs
Programmable Calculators
Probability Areas

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The report contains program listings and user instructions for an HP-41CV, a Sharp PC-1500 (or TRS-80 PC-2), a Sharp PC-1211 (or TRS-80 PC-1), a Casio FX-702P and a TI-59. The programs implement a bearings-only position estimation procedure. A development for the procedure is included in the report.

The programs in this report are for use
within the Department of the Navy, and they are
presented without representation or warranty of
any kind.

TABLE OF CONTENTS

I. Introduction

This report contains a target position estimation program for each of the following calculators: The Hewlett-Packard HP-41CV, the Sharp PC-1500 (or Radio Shack TRS-80 PC-2), the Sharp PC-1211 (or Radio Shack TRS-80 PC-1), the Casio FX-702P and the Texas Instruments TI-59.

The programs provide a means of implementing a position estimation procedure that is described in Appendix 1. The procedure is based on the following assumptions: Bearings taken on or from a target are available from two or more stations of known position. The positions of the stations and the target are such that they can be considered to be on the surface of a plane (a flat earth model). The error in a bearing taken on or from a station is a normal random variable. Its standard deviation (bearing error) is known and its mean (bias) is zero (if there is bias in a bearing, it is known and it is removed). The errors in bearing measurements are independent.

The position estimation procedure requires an initial estimate of the target's position. In the programs that are given here, the initial estimate is at the intersection of the bearing lines determined by the first two target bearings that are input to the program. Because of the use of this method to choose an initial estimate, the first two stations, with respect to the order of data input, should be chosen so that the intersection of their bearing lines is likely to be closer to the target's position than the intersection of the bearing lines from any other pair of stations.

1

The target ranges and errors of the bearings of the first two stations determine the distance between the intersection of their bearing lines and the target's position. In particular, if the angular separation between two stations as seen from the target is small relative to the bearing error of one or both of the stations, the bearing lines from the two stations may not intersect. If they do not intersect and they are not parallel, their reciprocal bearing lines will. In this case, if the observed bearings from the two stations were the first two bearing inputs to the program, the initial estimate of the target's position would be at the intersection of the two reciprocal bearing lines, and a gross error in the final position estimate could result.

II. Program User Instructions

Before using a program for the first time, refer to the notes that follow the user's instructions. Also, note the comments in Section I regarding the relationship between the accuracy of the computed estimates and the order of data entry.

For a PC-1211, use the PC-1500 User Instructions. Use the DEF mode and substitute SHFT for DEF wherever DEF appears in the instructions.

Station positions are determined with respect to a known reference point. The reference point can be a station position, in this case the station bearing and range from the reference point are both zero.

Program listings are given in Section IV and two example applications are discussed in Section III.

Position Estimation Program

HP-41CV User Instructions

Step	Instruction	Prompt	Press
1	Select USER mode and enter the program (see Note 1). Press Σ+ to run the program.		Σ+
2	Key in, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	OBS BRG?	R/S
3	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	R/S
4	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	R/S
5	Key in the bearing error (standard deviation) in decimal degrees.		R/S
6	Repeat Steps 2, 3, 4 & 5 for one or more additional stations.		
7	Compute bearing & range estimates.		1/x
8	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	R/S
9	A computed target range estimate in meters. For an elliptical containment region, go to Step 10 or Step 17. To enter additional data from new or old stations, go to Step 24.	RNG=est.	
10	For a containment ellipse of a given containment probability, press LN.		LN
11	Key in the desired containment probability.	PRB?	R/S
12	Computed value of the ellipse size: (See Note 3.)	SIZE=val.	R/S

Step	Instruction	Prompt	Press
13	Computed semi-major axis length:	SMJ=val.	R/S
14	Computed major axis direction:	DRC=val.	R/S
15	Computed semi-minor axis length:	SMI=val.	R/S
16	Computed containment ellipse area: (See Note 4.)	A = val.	
17	For a containment ellipse of a given size, press LOG.		LOG
18	Key in the desired containment ellipse size (see Note 3).	SIZE?	R/S
19	Computed value of the containment probability:	PRB=val.	R/S
20	Computed semi-major axis length:	SMJ=val.	R/S
21	Computed major axis direction:	DRC=val.	R/S
22	Computed semi-minor axis length:	SMI=val.	R/S
23	Computed containment ellipse area: (See Note 4.)	A = val.	
24	To enter additional data from new or old stations, press √x and then repeat Steps 2, 3, 4 & 5		√x̄

Notes:

1. The program size is 35. The key assignments are: TPE → Σ+, CON → √x̄, EST → 1/x, SIZ → LOG and PRB → LN. If they are not present, they must be made in order to follow the user instructions. For an alternative, see Appendix 2.

2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).

3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis $= k\sigma_{MJ}$ and semi-minor axis $= k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.

4. The area units are the range units squared.

Position Estimation Program

PC-1500 User Instructions

Step	Instruction	Prompt	Press
1	Enter the program. To run the program, press DEF, A. For a PC-1211, see Note 5.		DEF A
2	Key in, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 1).	OBS BRG?	ENTER
3	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	ENTER
4	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	ENTER
5	Key in the bearing error (standard deviation) in decimal degrees.	BRG ERR?	ENTER
6	Repeat Steps 2, 3, 4 & 5 for one or more additional stations.		
7	Compute bearing & range estimates.		DEF Z
8	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	ENTER
9	A computed target range estimate is displayed. For an elliptical containment region, go to Step 10 or Step 17. To enter additional data from new or old stations, go to Step 24.	RNG=est.	
10	For a containment ellipse of a given containment probability, press DEF, X.		DEF X
11	Key in the desired containment probability.	PRB?	ENTER
12	Computed value of the ellipse size: See Note 2.	SIZE=val.	ENTER

7

Step	Instruction	Prompt	Press
13	Computed semi-major axis length:	SMJ=val.	ENTER
14	Computed major axis direction:	DRC=val.	ENTER
15	Computed semi-minor axis length:	SMI=val.	ENTER
16	Computed containment ellipse area: (See Note 3.)	A = val.	
17	For a containment ellipse of a given size, press DEF, S.		DEF S
18	Key in the desired containment ellipse size (see Note 2).	SIZE?	ENTER
19	Computed value of the containment probability:	PRB=val.	ENTER
20	Computed semi-major axis length:	SMJ=val.	ENTER
21	Computed major axis direction:	DRC=val.	ENTER
22	Computed semi-minor axis length:	SMI=val.	ENTER
23	Computed containment ellipse area: (See Note 3.)	A = val.	
24	To enter additional data from new or old stations, press DEF, C and then repeat Steps 2, 3, 4 & 5.		DEF C

Notes:

1. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).

2. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis $= k\sigma_{MJ}$ and semi-minor axis $= k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.

3. The area units are the range units squared.

4. For a definition of the program initiating keys and their function, press DEF, H. The display will show:
   TPE = A EST = Z SIZ = S PRB = X. Next press ENTER. The display will show: CON = C. To repeat the display, press ENTER.

5. For a PC-1211, use the DEF mode and substitute SHFT for DEF wherever DEF appears in the instructions.

Position Estimation Program

FX-702P User Instructions

Step	Instruction	Prompt	Press
1	Enter the program (see Note 1). To run the program, first press F1, Ø.		F1 Ø
2	Next, key in 1 and press EXE.	OPTION?	1 EXE
3	Key in, in decimal degress, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	OBS BRG?	EXE
4	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	EXE
5	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	EXE
6	Key in the bearing error (standard deviation) in decimal degrees.	BRG ERR?	EXE
7	Repeat Steps 3, 4, 5 & 6 for one or more additional stations.		
8	To compute bearing & range estimates, first press F1, Ø.		F1 Ø
9	Next, key in 2 and press EXE.	OPTION?	2 EXE
10	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	CONT
11	A computed target range estimate is displayed. For an elliptical containment region, go to Step 12 or Step 20. To enter additional data from new or old stations, go to Step 28.	RNG=est.	
12	For a containment ellipse of a given containment probability, first press F1, Ø.		F1 Ø
13	Next, key in 3 and press EXE.	OPTION?	3 EXE

10

Step	Instruction	Prompt	Press
14	Key in the desired probability.	PRB?	EXE
15	Computed value of the ellipse size: See Note 3.	SIZE=val.	CONT
16	Computed semi-major axis length:	SMJ=val.	CONT
17	Computed major axis direction:	DRC-val.	CONT
18	Computed semi-minor axis length:	SMI=val.	CONT
19	Computed containment ellipse area: See Note 4.	A = val.	
20	For a containment ellipse of a given size, first press F1, Ø.		F1 Ø
21	Next, key in 4 and press EXE.	OPTION?	4 EXE
22	Key in the desired containment ellipse size (see Note 3).	SIZE?	EXE
23	Computed value of the containment probability:	PRB=val.	CONT
24	Computed semi-major axis length:	SMJ=val.	CONT
25	Computed major axis direction:	DRC=val.	CONT
26	Computed semi-minor axis length:	SMI=val.	CONT
27	Computed containment ellipse area: See Note 4.	A = val.	
28	To enter additional data from new or old stations, first press F1, Ø.		F1 Ø
29	Next, key in 5, press EXE and then repeat Steps 3, 4, 5 & 6.	OPTION?	5 EXE

Notes:

1. Enter the program in P$\emptyset$ for F1 $\emptyset$ activation. Before running the program, first press F2 then DEFM then 1 and then EXE. This is required in order to use the array variables in the program.

2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).

3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis $= k\sigma_{MJ}$ and semi-minor axis $= k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.

4. The area units are the range units squared.

5. For a definition of the program options and their function, press F1, $\emptyset$. Then after OPTION? is displayed, enter $\emptyset$ and press EXE. The display will show: TPE = 1 EST = 2 SIZ = 3. Next, press CONT. The display will show: PRB = 4 CON = 5. To repeat the displays, press CONT.

POSITION ESTIMATION PROGRAM

TI-59 USER INSTRUCTIONS

Step	Instruction	Enter	Press	Display
1	Enter the program (see Note 1).			
2	To run the program, press A.		A	9
3	Enter, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	θ	B	θ
4	Enter the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	α	R/S	α
5	Enter the station range from the reference point. Use zero if the station is the reference point. Use any units.	ρ	R/S	ρ
6	Enter the bearing error (standard deviation) in decimal degrees. After pressing R/S, the display indicates the order number of the data entry.	e	R/S	n
7	Repeat Steps 3, 4, 5 & 6 for at least one more station.			
8	Display a bearing estimate $\hat{\phi}$ with respect to the reference station.		C	$\hat{\phi}$
9	Display a range estimate $\hat{r}$ with respect to the reference station. For an elliptical containment region, go to Step 10 or Step 15. To enter additional data from new or old stations, repeat Steps 3, 4, 5 & 6.		R/S	$\hat{r}$
10	For a containment ellipse of a given containment probability, enter the containment probability. Next, press E and display the ellipse size (see Note 3).	p	E	k
11	Display the semi-major axis length.		R/S	$k\sigma_{MJ}$
12	Display the major axis direction.		R/S	γ
13	Display the semi-minor axis length.		R/S	$k\sigma_{MI}$

Step	Instruction	Enter	Press	Display
14	Display the containment ellipse area. (See Note 4.)		R/S	area
15	For a containment ellipse of a given size, enter the containment ellipse size (See Note 3). Next, press D and display the containment probability.	k	D	p
16	Display the semi-major axis length.		R/S	$k\sigma_{MJ}$
17	Display the major axis direction.		R/S	γ
18	Display the semi-minor axis length.		R/S	$k\sigma_{MI}$
19	Display the containment ellipse area. (See Note 4.)		R/S	area
20	To enter additional data from new or old stations, repeat Steps 3, 4, 5 & 6.			

Notes:

1. The program requires the normal partition. If the calculator has been in use, this can be assured by turning the calculator off and then on before loading the program.

2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).

3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis $= k\sigma_{MJ}$ and semi-minor axis $= k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.

4. The area units are the range units squared.

5. Negative bearing estimates and negative major axis directions can result. To convert a negative bearing estimate to the value that would be output by the other programs, add 360° to the estimate. For example, -5° becomes 355°. To convert a negative direction, add 180°. For example, -5° becomes 175°.

III. Two Examples

In Scenario 1, the scenario for the first example, bearings are taken on a target from three separate stations. Figure 1 on Page 18 shows Scenario 1 and Table 1 below gives the station data. The stations are numbered according to the order of station data input to the program.

TABLE 1

	OBS BRG	STA BRG	STA RNG	BRG ERR
Station 1	038°	334°	13500	4°
Station 2	324°	050°	11350	3°
Station 3	003°	000°	0	4°

Note from Table 1 that the reference point is at Station 3.

Program outputs for Scenario 1 are given in List 1 on Page 19. List 1 and List 2 which gives program outputs for Scenario 2, are copies of printer tapes that were generated with a Casio FP-10 printer and a Casio FX-702P using the Casio FX-702P program. With allowance for differences in user instructions, display formats and round-off errors, the tapes indicate the output that should be obtained using any of the other calculator programs except for the TI-59 program where equivalent negative angles occur.

In Scenario 2, the scenario for the second example, bearings are taken from a target on three separate stations. Figure 2 on Page 20 shows Scenario 2 and Table 2 below gives the station data. As in Scenario 1, the stations are numbered according to the order of station data input to the program. In this scenario the reciprocal of the observed bearings are used in order to provide an equivalent scenario that is appropriate for the program.

TABLE 2

	OBS BRG	RCP BRG	STA BRG	STA RNG	BRG ERR
Station 1	211°	031°	000°	0	3°
Station 2	172°	352°	115°	11100	3°
Station 3	146°	326°	082°	13800	3°

Note from Table 2 that the reference point is at Station 1.

Program outputs for Scenario 2 are given in List 2 on Page 21. The two bearing estimates in the list are the reciprocal of the target's estimate of the bearing of Station 1.
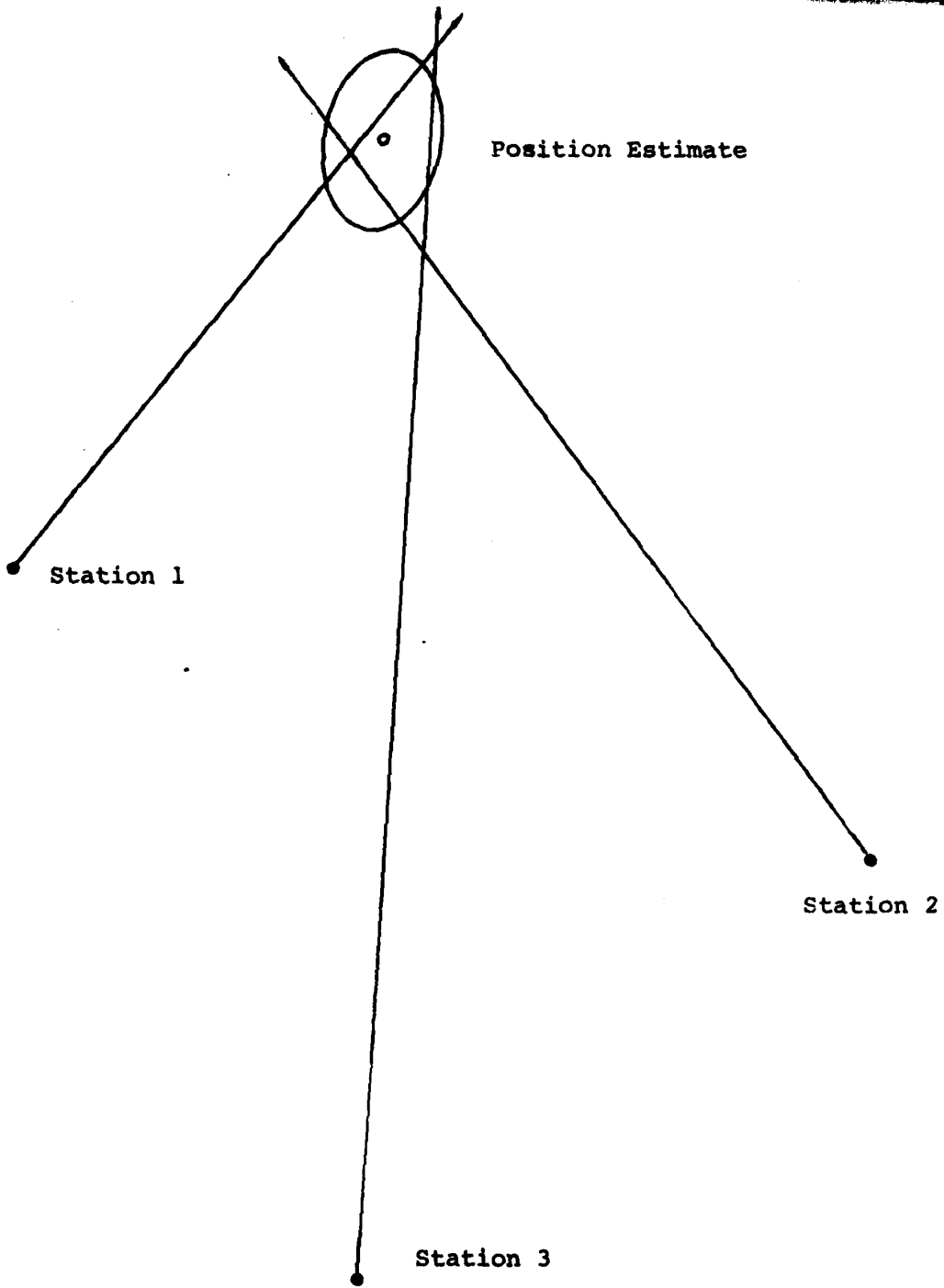
Figure 1. A position estimate and a .83 confidence region.
Table 1 gives the bearings of the indicated bearing
lines and List 1 gives the bearing and the range of
the position estimate from Station 3.

18

```
OPTION?                          OPTION?
0                                5
TPE=1 EST=2 SIZ=3                OBS BRG?
                                 3
PRB=4 CON=5                      STA BRG?
                                 0
OPTION?                          STA RNG?
1                                0
OBS BRG?                         BRG ERR?
38                               4
STA BRG?                         OBS BRG?
334
STA RNG?                         OPTION?
13500                            2
BRG ERR?                         BRG= 0.15
4
OBS BRG?                         RNG= 19553.78
324
STA BRG?                         END
50
STA RNG?                         OPTION?
11350                            3
BRG ERR?                         SIZE?
3                                2
OBS BRG?                         PRB= 0.86

OPTION?                          SMJ= 1712.95
2
BRG= 359.51                      DIR= 12.48

RNG= 19494.39                    SMI= 1129.90

END                              A= 6000418.40

OPTION?                          END
3
SIZE?                            OPTION?
2                                4
PRB= 0.86                        PRB?
                                 .9
SMJ= 1737.32                     SIZE= 2.15

DIR= 17.69                       SMJ= 1837.97

SMI= 1232.96                     DIR= 12.48

A= 6729444.91                    SMI= 1212.36

END                              A= 7000340.38

OPTION?                          END
4
PRB?
.9
SIZE= 2.15

SMJ= 1864.12

DIR= 17.69

SMI= 1322.94

A= 7747559.77

END
```
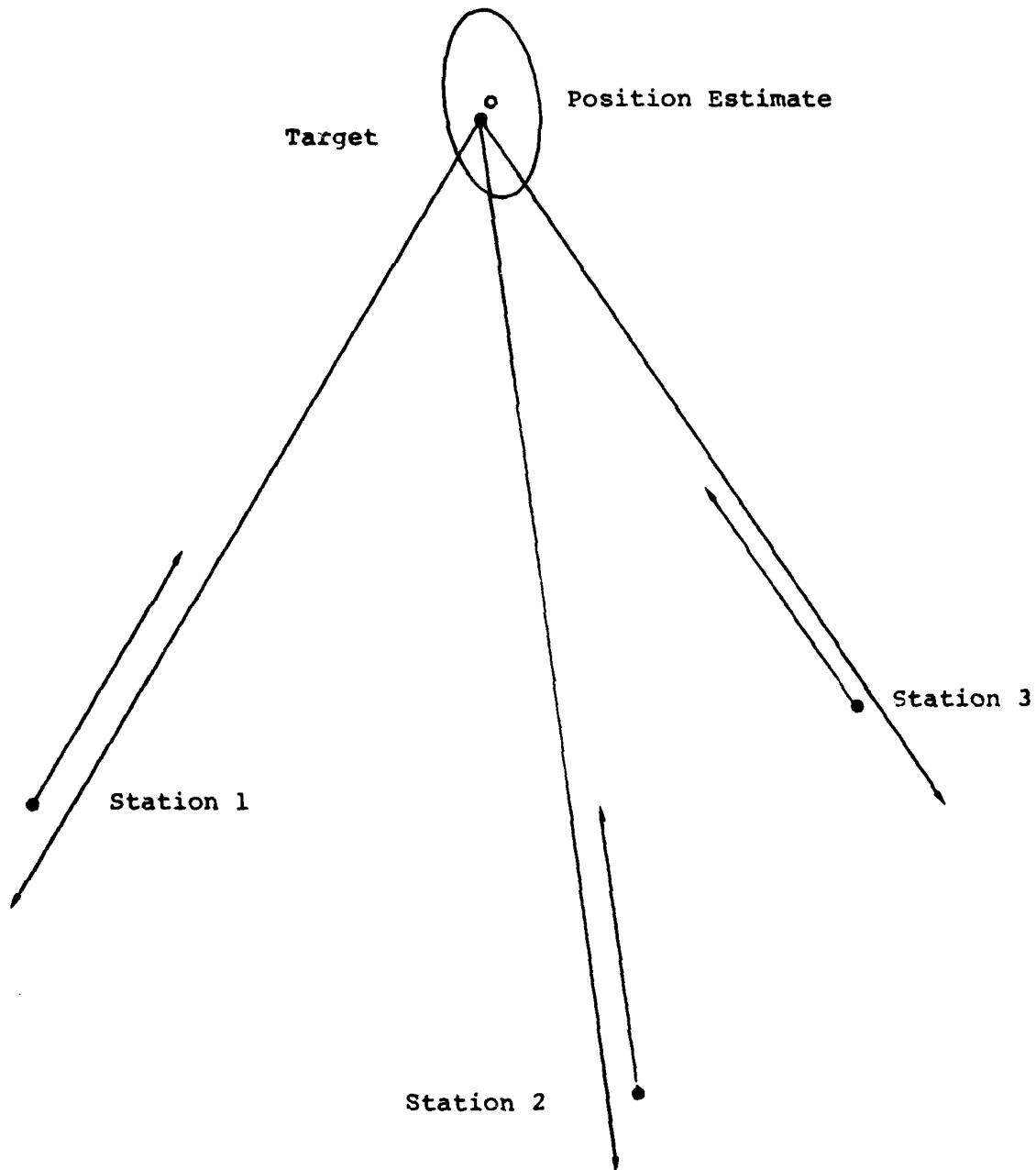
Figure 2. A position estimate and .71 confidence region.
Table 2 gives the bearings of the indicated bearing
lines and List 2 gives the reciprocal bearing and
the range of the position estimate from Station 1.

OPTION?
0
TPE=1 EST=2 SIZ=3

PRB=4 CON=5

OPTION?
1
OBS BRG?
31
STA BRG?
0
STA RNG?
0
BRG ERR?
3
OBS BRG?
352
STA BRG?
115
STA RNG?
11100
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 31.00

RNG= 14792.53

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 3606.56

DIR= 15.39

SMI= 1253.72

A= 14205070.91

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 3869.77

DIR= 15.39

SMI= 1345.22

A= 16354192.26

END

OPTION?
5
OBS BRG?
326
STA BRG?
82
STA RNG?
13800
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 31.65

RNG= 13765.30

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 2023.19

DIR= 172.06

SMI= 967.90

A= 6151983.32

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 2170.85

DIR= 172.06

SMI= 1038.54

A= 7082732.54

END

21

IV. Program Listings

```
01♦LBL "TPE"        51 -                101 ST+ 10
02 DEG              52 SIN              102 RCL IND 10
03 CLRG            53 *                103 STO 12
04 .001            54 STO 15            104 1
05 STO 09          55 RCL 33            105 ST+ 10
06 26              56 RCL 32            106 RCL IND 10
07 STO 10          57 RCL 31            107 STO 13
08♦LBL "CON"        58 -                108 1
09 FIX 0           59 SIN              109 ST+ 10
10 "OBS BRG?"       60 *                110 RCL IND 10
11 PROMPT          61 STO 16            111 STO 14
12 STO 11          62 RCL 31            112 XEQ 07
13 "STA BRG?"       63 RCL 27            113 ISG 09
14 PROMPT          64 -                114 GTO 05
15 STO 12          65 SIN              115 GTO "CON"
16 "STA RNG?"       66 STO 17            116♦LBL 02
17 PROMPT          67 X=0?              117 XEQ 07
18 STO 13          68 GTO 08            118 GTO "CON"
19 "BRG ERR?"       69 RCL 15            119♦LBL 07
20 PROMPT          70 RCL 31            120 RCL 01
21 STO 14          71 SIN              121 RCL 13
22 1               72 *                122 RCL 12
23 ST+ 08          73 RCL 16            123 SIN
24 2               74 RCL 27            124 *
25 RCL 08          75 SIN              125 -
26 X>Y?            76 *                126 RCL 02
27 GTO 02          77 -                127 RCL 13
28 1               78 RCL 17            128 RCL 12
29 ST+ 10          79 /                129 COS
30 RCL 11          80 STO 01            130 *
31 STO IND 10      81 RCL 15            131 -
32 1               82 RCL 31            132 R-P
33 ST+ 10          83 COS              133 STO 16
34 RCL 12          84 *                134 X<>Y
35 STO IND 10      85 RCL 16            135 STO 15
36 1               86 RCL 27            136 RCL 11
37 ST+ 10          87 COS              137 RCL 15
38 RCL 13          88 *                138 -
39 STO IND 10      89 -                139 180
40 1               90 RCL 17            140 X<>Y
41 ST+ 10          91 /                141 X<=Y?
42 RCL 14          92 STO 02            142 GTO 11
43 STO IND 10      93 26               143 360
44 1               94 STO 10            144 -
45 RCL 08          95♦LBL 05            145♦LBL 11
46 X=Y?            96 1                146 RCL 14
47 GTO "CON"        97 ST+ 10            147 /
48 RCL 29          98 RCL IND 10        148 STO 17
49 RCL 28          99 STO 11            149 RCL 16
50 RCL 27          100 1               150 RCL 14
```

151 *
152 PI
153 *
154 180
155 /
156 STO 18
157 RCL 15
158 COS
159 RCL 18
160 /
161 STO 19
162 RCL 15
163 SIN
164 RCL 18
165 /
166 STO 20
167 RCL 19
168 X↑2
169 ST+ 03
170 RCL 19
171 RCL 20
172 *
173 ST+ 04
174 RCL 20
175 X↑2
176 ST+ 05
177 RCL 17
178 RCL 19
179 *
180 ST+ 06
181 RCL 17
182 RCL 20
183 *
184 ST+ 07
185 RTN
186*LBL "EST"
187 FIX 2
188 RCL 04
189 X↑2
190 RCL 03
191 RCL 05
192 *
193 -
194 STO 20
195 X=0?
196 GTO 08
197 RCL 01
198 RCL 04
199 RCL 07
200 *

201 RCL 05
202 RCL 06
203 *
204 -
205 RCL 20
206 /
207 +
208 STO 26
209 RCL 02
210 RCL 03
211 RCL 07
212 *
213 RCL 04
214 RCL 06
215 *
216 -
217 RCL 20
218 /
219 +
220 RCL 26
221 X<>Y
222 R-P
223 STO 21
224 X<>Y
225 X>0?
226 GTO 06
227 360
228 +
229*LBL 06
230 STO 22
231 RCL 04
232 RCL 03
233 RCL 05
234 X≠Y?
235 GTO 03
236 RCL 04
237 SIGN
238 45
239 *
240 GTO 04
241*LBL 03
242 -
243 /
244 2
245 *
246 ATAN
247 2
248 /
249*LBL 04
250 STO 23

251 COS
252 X↑2
253 RCL 05
254 *
255 RCL 23
256 1
257 P-R
258 *
259 2
260 *
261 RCL 04
262 *
263 STO 26
264 -
265 RCL 23
266 SIN
267 X↑2
268 RCL 03
269 *
270 +
271 RCL 20
272 CHS
273 /
274 STO 24
275 RCL 23
276 SIN
277 X↑2
278 RCL 05
279 *
280 RCL 26
281 +
282 RCL 23
283 COS
284 X↑2
285 RCL 03
286 *
287 +
288 RCL 20
289 CHS
290 /
291 STO 25
292 RCL 24
293 X<=Y?
294 GTO 01
295 STO 25
296 X<>Y
297 STO 24
298 90
299 ST+ 23
300*LBL 01

24

```
301 "BRG="          351 *
302 ARCL 22         352 RCL 11
303 AVIEW           353 RCL 24
304 STOP            354 SQRT
305 "RNG="          355 *
306 ARCL 21         356 X<>Y
307 AVIEW           357 "SMJ="
308 STOP            358 ARCL X
309 "END"           359 AVIEW
310 GTO 00          360 STOP
311*LBL "SIZ"       361 RCL 23
312 "SIZE"          362 X>0?
313 PROMPT          363 GTO 10
314 STO 11          364 180
315 X↑2             365 +
316 2               366*LBL 10
317 /               367 "BRC="
318 CHS             368 ARCL X
319 E↑X             369 AVIEW
320 1               370 STOP
321 X<>Y            371 "SMI="
322 -               372 RDN
323 STO 00          373 ARCL Y
324 "PRB="          374 AVIEW
325 ARCL 00         375 STOP
326 AVIEW           376 PI
327 STOP            377 *
328 GTO 09          378 *
329*LBL "PRB"       379 "A="
330 "PRB?"          380 ARCL X
331 PROMPT          381*LBL 00
332 CHS             382 AVIEW
333 1               383 STOP
334 +               384 "END"
335 LN              385 GTO 00
336 2               386*LBL 08
337 *               387 AVIEW
338 CHS             388 STOP
339 SQRT            389 "NO SOL"
340 STO 11          390 GTO 08
341 "SIZE="         391 END
342 ARCL 11
343 AVIEW
344 STOP
345*LBL 09
346 RCL 23
347 STO 12
348 RCL 11
349 RCL 25
350 SQRT
```

```
10:"A":CLEAR :DIM
   A(7):DEGREE
15:"C":INPUT "OBS
   BRG? ";P:
   INPUT "STA BRG
   ? ";Q:INPUT "S
   TA RNG? ";R:
   INPUT "BRG ERR
   ? ";O:PAUSE "
   "
20:IF I=2GOTO 50
25:I=I+1:A(I-1)=P
   :A(I+1)=Q:A(I+
   3)=R:A(I+5)=O:
   IF I=1GOTO 15
30:X=A(4)*SIN (A(
   2)-A(0)):Y=A(5
   )*SIN (A(3)-A(
   1)):Z=SIN (A(1
   )-A(0)):IF Z=0
   GOTO 150
35:U=(X*SIN A(1)-
   Y*SIN A(0))/Z:
   U=(X*COS A(1)-
   Y*COS A(0))/Z
40:FOR M=0TO 1
45:P=A(M):Q=A(M+2
   ):R=A(M+4):O=A
   (M+6):GOSUB 17
   0:NEXT M:GOTO
   15
50:GOSUB 170:GOTO
   15
60:"Z":PAUSE " ":
   F=(B*B-A*C):IF
   F=0GOTO 150
65:X=U+(B*E-C*D)/
   F:Y=U+(A*E-B*D
   )/F:GOSUB 200
70:T=SGN B*45:IF
   A=CGOTO 80
75:T=.5*ATN (2*B/
   (A-C))
80:G=(C*COS T*COS
   T-2*B*COS T*
   SIN T+A*SIN T*
   SIN T)/-F:G=√G
85:H=(C*SIN T*SIN
   T+2*B*COS T*
   SIN T+A*COS T*
   COS T)/-F:H=√H
   :IF H>=GGOTO 9
   5
90:Z=H:H=G:G=Z:T=
   T+90
95:PRINT "BRG=";J
   :PRINT "RNG=";
   K:GOTO 145
100:"H":PRINT " TP
   E=A EST=Z SIZ=
   S PRB=X"
105:PRINT "CON=C":
   GOTO 100
120:"S":INPUT "SIZ
   E? ";S:O=1-EXP
   (-S*S/2)
125:PRINT USING "#
   #.##";"PRB=";O
   :USING :GOTO 1
   35
130:"X":INPUT "PRB
   ? ";O:S=√(-2*
   LN (1-O)):
   PRINT "SIZE=";
   S
135:X=S*G:Y=S*H:N=
   T:PRINT "SMJ="
   ;Y:IF N<0LET N
   =N+180
140:PRINT "DIR=";N
   :PRINT "SMI=";
   X:PRINT "A=";π
   *X*Y
145:PRINT "END":
   GOTO 145
150:PRINT "NO SOL"
   :GOTO 150
170:X=U-R*SIN Q:Y=
   U-R*COS Q:
   GOSUB 200
175:W=(P-J):L=K*O*
   π/180:G=COS J/
   L:H=SIN J/L:IF
   W>=180LET W=W-
   360:GOTO 185
180:IF W<=-180LET
   W=W+360
185:W=W/O:A=G*G+A:
   B=G*H+B:C=H*H+
   C:D=W*G+D:E=W*
   H+E:RETURN
200:K=√(X*X+Y*Y):
   IF K=0LET J=0:
   RETURN
205:J=ACS (Y/K):IF
   ASN (X/K)<0LET
   J=360-J
210:RETURN
```

```
10:"A":CLEAR :
   DEGREE
15:"C":INPUT "O
   BS BRG? ";P:
   INPUT "STA B
   RG? ";Q:
   INPUT "STA R
   NG? ";R:
   INPUT "BRG E
   RR? ";O:IF I
   =2GOTO 130
75:I=I+1:A(I+26
   )=P:A(I+28)=
   Q:A(I+30)=R:
   A(I+32)=O:IF
   I=1GOTO 15
85:X=A(31)*SIN
   (A(29)-A(27)
   ):Y=A(32)*
   SIN (A(30)-A
   (28)):Z=SIN
   (A(28)-A(27)
   )
90:IF Z=0GOTO 3
   25
95:U=(X*SIN A(2
   8)-Y*SIN A(2
   7))/Z:V=(X*
   COS A(28)-Y*
   COS A(27))/Z
105:FOR I=1TO 2:
   P=A(I+26):Q=
   A(I+28):R=A(
   I+30):O=A(I+
   32):GOSUB 40
   0
110:NEXT I:GOTO
   15
130:GOSUB 400
135:GOTO 15
140:"Z":F=(B*B-A
   *C):IF F=0
   GOTO 325
150:X=U+(B*E-C*D
   )/F:Y=V+(A*E
   -B*D)/F:
   GOSUB 500
160:T=SGN B*45:
   IF A=CTHEN 1
   70
165:T=.5*ATN (2*
   B/(A-C))
170:G=(C*COS T*
   COS T-2*B*
   COS T*SIN T+
   A*SIN T*SIN
   T)/-F:G=√G
175:H=(C*SIN T*
   SIN T+2*B*
   COS T*SIN T+
   A*COS T*COS
   T)/-F:H=√H:
   IF H>=GGOTO
   185
180:Z=H:H=G:G=Z:
   T=T+90
185:PRINT "BRG="
   ;J:PRINT "RN
   G=";K:GOTO 3
   20
200:"H":PRINT "T
   PE=A EST=Z S
   IZ=S PRB=X"
205:PRINT "CON=C
   ":GOTO 200
225:"S":INPUT "S
   IZE? ";S:O=1
   -EXP (-S*S/2
   )
230:PRINT USING
   "##.##";"PRB
   =";O:USING :
   GOTO 300
235:"X":INPUT "P
   RB? ";O:S=√(
   -2*LN (1-O))
   :PRINT "SIZE
   =";S
300:X=S*G:Y=S*H:
   N=T:PRINT "S
   MJ=";Y:IF N
   <=0LET N=N+1
   80
315:PRINT "DRC="
   ;N:PRINT "SM
   I=";X:Z=π*X*
   Y:PRINT "A="
   ;Z
320:PRINT "END":
   GOTO 320
325:PRINT "NO SO
   L":GOTO 325
400:X=U-R*SIN Q:
   Y=V-R*COS Q:
   GOSUB 500:W=
   (P-J):L=K*O*
   π/180:G=COS
   J/L:H=SIN J/
   L
415:IF W>=180LET
   W=W-360:GOTO
   440
425:IF W<=-180
   LET W=W+360
440:W=W/O:A=G*G+
   A:B=G*H+B:C=
   H*H+C:D=W*G+
   D:E=W*H+E:
   RETURN
500:K=√(X*X+Y*Y)
   :IF K=0LET J
   =0:RETURN
510:M=ASN (X/K):
   J=ACS (Y/K):
   IF M<0LET J=
   360-J
515:RETURN
```

```
5 INP "OPTION",J:
  SET F2:MODE 4
10 IF J=0 THEN 175
15 IF J=2 THEN 100
20 IF J=3 THEN 145
25 IF J=4 THEN 150
30 IF J=5 THEN 45
40 VAC :SET F2
45 INP "OBS BRG",P
  :INP "STA BRG",
  Q:INP "STA RNG"
  ,R
50 INP "BRG ERR",O
  :IF I=2 THEN 95
55 I=I+1:A(I-1)=P:
  A(I+1)=Q:A(I+3)
  =R:A(I+5)=O
60 IF I=1 THEN 45
65 X=A(4)*SIN (A(2
  )-A(0)):Y=A(5)*
  SIN (A(3)-A(1))
70 Z=SIN (A(1)-A(0
  )):IF Z=0 THEN
  170
75 U=(X*SIN A(1)-Y
  *SIN A(0))/Z:V=
  (X*COS A(1)-Y*C
  OS A(0))/Z
80 FOR M=0 TO 1:P=
  A(M):Q=A(M+2)
85 R=A(M+4):O=A(M+
  6):GSB 200:NEXT
  M:GOTO 45
90 R=A(M+4):O=A(M+
  6):GSB 200:NEXT
  M:GOTO 45
95 GSB 200:GOTO 45
100 F=(B*B-A*C):IF
  F=0 THEN 170
105 X=U+(B*E-C*D)/F
  :Y=V+(A*E-B*D)/
  F:RPC Y,X
110 T=SGN B*45:IF A
  =C THEN 120
115 T=.5*ATN (2*B/(
  A-C))
120 G=((C*COS T*COS
  T-2*B*COS T*SIN
  T+A*SIN T*SIN
  T)/-F
```

```
125 H=((C*SIN T*SIN
  T+2*B*COS T*SIN
  T+A*COS T*COS
  T)/-F
130 G=SQR G:H=SQR H
  :IF G>H:Z=H:H=G
  :G=Z:T=T+90
135 IF Y<0:Y=Y+360
140 PRT "BRG=";Y:PR
  T "RNG=";X:GOTO
  165
145 INP "SIZE",S:O=
  1-EXP (-S*S/2):
  PRT "PRB=";O:GO
  TO 155
150 INP "PRB",O:S=S
  QR (-2*LN (1-O)
  ):PRT "SIZE=";S
155 X=S*G:Y=S*H:PRT
  "SMJ=";Y:N=T:I
  F N<0:N=N+180
160 PRT "DIR=";N:PR
  T "SMI=";X:PRT
  "A=";π*X*Y
165 PRT "END":GOTO
  165
170 PRT "NO SOL":GO
  TO 170
175 PRT "TPE=1 EST=
  2 SIZ=3"
180 PRT "PRB=4 CON=
  5":GOTO 175
200 X=U-R*SIN Q:Y=V
  -R*COS Q:RPC Y,
  X
205 W=(P-Y):L=X*O*π
  /180:G=COS Y/L:
  H=SIN Y/L
210 IF W≥180:W=W-36
  0:GOTO 220
215 IF W≤-180:W=W+3
  60
220 W=W/O:A=G*G+A:B
  =G*H+B:C=H*H+C:
  D=W*G+D:E=W*H+E
  :RET
```

000	47	CMS	050	10	E'	100	42	STO
001	60	DEG	051	18	C'	101	38	38
002	02	2	052	87	IFF	102	43	RCL
003	42	STO	053	03	03	103	14	14
004	09	09	054	01	01	104	39	COS
005	69	OP	055	28	28	105	65	×
006	28	28	056	19	D'	106	43	RCL
007	76	LBL	057	75	-	107	18	18
008	16	A'	058	19	D'	108	75	-
009	09	9	059	95	=	109	43	RCL
010	42	STO	060	94	+/-	110	10	10
011	00	00	061	38	SIN	111	39	COS
012	42	STO	062	65	×	112	65	×
013	01	01	063	19	D'	113	43	RCL
014	92	RTN	064	95	=	114	19	19
015	76	LBL	065	48	EXC	115	95	=
016	18	C'	066	19	19	116	55	÷
017	69	OP	067	22	INV	117	43	RCL
018	20	20	068	97	DSZ	118	20	20
019	72	ST*	069	09	09	119	38	SIN
020	00	00	070	00	00	120	95	=
021	92	RTN	071	76	76	121	42	STO
022	76	LBL	072	01	1	122	39	39
023	19	D'	073	69	OP	123	86	STF
024	69	OP	074	21	21	124	03	03
025	21	21	075	92	RTN	125	02	2
026	73	RC*	076	42	STO	126	42	STO
027	01	01	077	18	18	127	09	09
028	92	RTN	078	65	×	128	16	A'
029	76	LBL	079	43	RCL	129	19	D'
030	10	E'	080	14	14	130	10	E'
031	65	×	081	42	STO	131	42	STO
032	89	π	082	20	20	132	18	18
033	55	÷	083	38	SIN	133	19	D'
034	01	1	084	75	-	134	32	X:T
035	08	8	085	43	RCL	135	19	D'
036	00	0	086	10	10	136	32	X:T
037	95	=	087	22	INV	137	37	P/R
038	92	RTN	088	44	SUM	138	75	-
039	76	LBL	089	20	20	139	43	RCL
040	11	A	090	38	SIN	140	38	38
041	81	RST	091	65	×	141	95	=
042	76	LBL	092	43	RCL	142	94	+/-
043	12	B	093	19	19	143	32	X:T
044	18	C'	094	95	=	144	75	-
045	91	R/S	095	55	÷	145	43	RCL
046	18	C'	096	43	RCL	146	39	39
047	91	R/S	097	20	20	147	95	=
048	18	C'	098	38	SIN	148	94	+/-
049	91	R/S	099	95	=	149	32	X:T

150	22	INV	200	49	PRD	250	43	43
151	37	P/R	201	26	26	251	42	STO
152	42	STO	202	49	PRD	252	10	10
153	19	19	203	27	27	253	65	×
154	10	E'	204	32	X:T	254	43	RCL
155	75	-	205	49	PRD	255	21	21
156	43	RCL	206	28	28	256	22	INV
157	18	18	207	49	PRD	257	44	SUM
158	95	=	208	26	26	258	10	10
159	94	+/-	209	33	X²	259	75	-
160	32	X:T	210	44	SUM	260	43	RCL
161	69	OP	211	21	21	261	22	22
162	21	21	212	43	RCL	262	22	INV
163	64	PD*	213	28	28	263	49	PRD
164	01	01	214	44	SUM	264	10	10
165	42	STO	215	22	22	265	33	X²
166	18	18	216	43	RCL	266	95	=
167	89	π	217	26	26	267	35	1/X
168	32	X:T	218	44	SUM	268	49	PRD
169	22	INV	219	24	24	269	41	41
170	77	GE	220	43	RCL	270	49	PRD
171	01	01	221	27	27	271	42	42
172	78	78	222	44	SUM	272	49	PRD
173	75	-	223	25	25	273	43	43
174	32	X:T	224	22	INV	274	02	2
175	65	×	225	97	DSZ	275	55	÷
176	02	2	226	09	09	276	43	RCL
177	95	=	227	02	02	277	10	10
178	49	PRD	228	32	32	278	95	=
179	18	18	229	61	GTO	279	24	CE
180	73	RC*	230	01	01	280	22	INV
181	01	01	231	29	29	281	30	TAN
182	35	1/X	232	69	OP	282	55	÷
183	42	STO	233	28	28	283	02	2
184	27	27	234	43	RCL	284	95	=
185	42	STO	235	08	08	285	42	STO
186	26	26	236	92	RTN	286	29	29
187	32	X:T	237	76	LBL	287	43	RCL
188	43	RCL	238	13	C	288	43	43
189	19	19	239	43	RCL	289	65	×
190	37	P/R	240	21	21	290	43	RCL
191	42	STO	241	42	STO	291	24	24
192	28	28	242	41	41	292	75	-
193	49	PRD	243	43	RCL	293	43	RCL
194	27	27	244	22	22	294	42	42
195	33	X²	245	42	STO	295	65	×
196	44	SUM	246	42	42	296	43	RCL
197	23	23	247	43	RCL	297	25	25
198	43	RCL	248	23	23	298	85	+
199	18	18	249	42	STO	299	43	RCL

300	38	38	350	14	14	400	01	1
301	95	=	351	43	RCL	401	95	=
302	32	X:T	352	42	42	402	94	+/-
303	43	RCL	353	65	×	403	23	LNX
304	42	42	354	02	2	404	65	×
305	65	×	355	95	=	405	02	2
306	43	RCL	356	49	PRD	406	95	=
307	24	24	357	11	11	407	94	+/-
308	75	-	358	43	RCL	408	34	ΓX
309	43	RCL	359	43	43	409	91	R/S
310	41	41	360	49	PRD	410	42	STO
311	65	×	361	13	13	411	15	15
312	43	RCL	362	49	PRD	412	65	×
313	25	25	363	15	15	413	43	RCL
314	85	+	364	43	RCL	414	16	16
315	43	RCL	365	15	15	415	95	=
316	39	39	366	85	+	416	42	STO
317	95	=	367	43	RCL	417	18	18
318	32	X:T	368	11	11	418	32	X:T
319	22	INV	369	85	+	419	43	RCL
320	37	P/R	370	43	RCL	420	29	29
321	42	STO	371	12	12	421	94	+/-
322	32	32	372	95	=	422	42	STO
323	01	1	373	34	ΓX	423	30	30
324	32	X:T	374	42	STO	424	43	RCL
325	42	STO	375	16	16	425	15	15
326	33	33	376	43	RCL	426	65	×
327	43	RCL	377	13	13	427	43	RCL
328	29	29	378	75	-	428	17	17
329	37	P/R	379	43	RCL	429	95	=
330	42	STO	380	11	11	430	77	GE
331	11	11	381	85	+	431	04	04
332	33	X²	382	43	RCL	432	41	41
333	42	STO	383	14	14	433	48	EXC
334	12	12	384	95	=	434	18	18
335	42	STO	385	34	ΓX	435	32	X:T
336	13	13	386	42	STO	436	09	9
337	32	X:T	387	17	17	437	00	0
338	49	PRD	388	43	RCL	438	44	SUM
339	11	11	389	32	32	439	30	30
340	33	X²	390	91	R/S	440	32	X:T
341	42	STO	391	43	RCL	441	91	R/S
342	14	14	392	33	33	442	32	X:T
343	42	STO	393	92	RTN	443	43	RCL
344	15	15	394	61	GTO	444	30	30
345	43	RCL	395	03	03	445	91	R/S
346	41	41	396	93	93	446	43	RCL
347	49	PRD	397	76	LBL	447	18	18
348	12	12	398	15	E	448	91	R/S
349	49	PRD	399	75	-	449	65	×

```
450    32  X:T
451    65   ×
452    89   π
453    95   =
454    92  RTN
455    61  GTO
456    04   04
457    54   54
458    76  LBL
459    14   D
460    33  X²
461    55   ÷
462    32  X:T
463    02   2
464    95   =
465    94  +/-
466    22  INV
467    23  LNX
468    75   -
469    01   1
470    95   =
471    94  +/-
472    91  R/S
473    32  X:T
474    34  √X
475    61  GTO
476    04   04
477    10   10
478    00   0
479    00   0
```

Appendix 1. The Estimation Procedure

In this development the assumptions stated in Section I are required conditions. A rectangular coordinate system is used with the positive y-axis directed north, the positive x-axis directed east and the origin at the reference point and all angles are in radians. Figure 3 shows a station located with respect to the coordinate system. The bearing line of length r goes to the object's unknown position, the bearing line of length r goes to an initial estimate of the object's position and the third bearing line corresponds to an observed bearing.

To determine the coordinates for a final estimate, consider the arc segments $u = r(\theta-\phi)$ where $\theta-\phi$ is the bearing error and $v = r(\phi-\beta)$ and $w = r(\theta-\beta)$ that are defined by the three bearing lines and the circle of radius r that is centered on the station and goes through the initial estimate. The geometry is shown in Figure 3. Note that u can be defined by $u = w - v$. In this expression, $w = r(\theta-\beta)$ is known, and v can be determined in terms of x and y the unknown coordinates of the target. With the reference point at the initial estimte, to first order, $v = x \cos \beta - y \sin \beta$ and $u = w - x \cos \beta + y \sin \beta$. Since this approximation applies to all stations, its use suggests that, for each station, $r \simeq r$ which is equivalent to having the initial estimate relatively close to the target's position.

Since, for each station i, an observed bearing θ_i is the value of a normal random variable Θ_i with mean ϕ_i, the coordinate $u_i = r_i(\theta_i-\phi_i)$ is the value of a normal random variable U_i with mean zero. In addition, since the Θ_i are independent, the U_i are
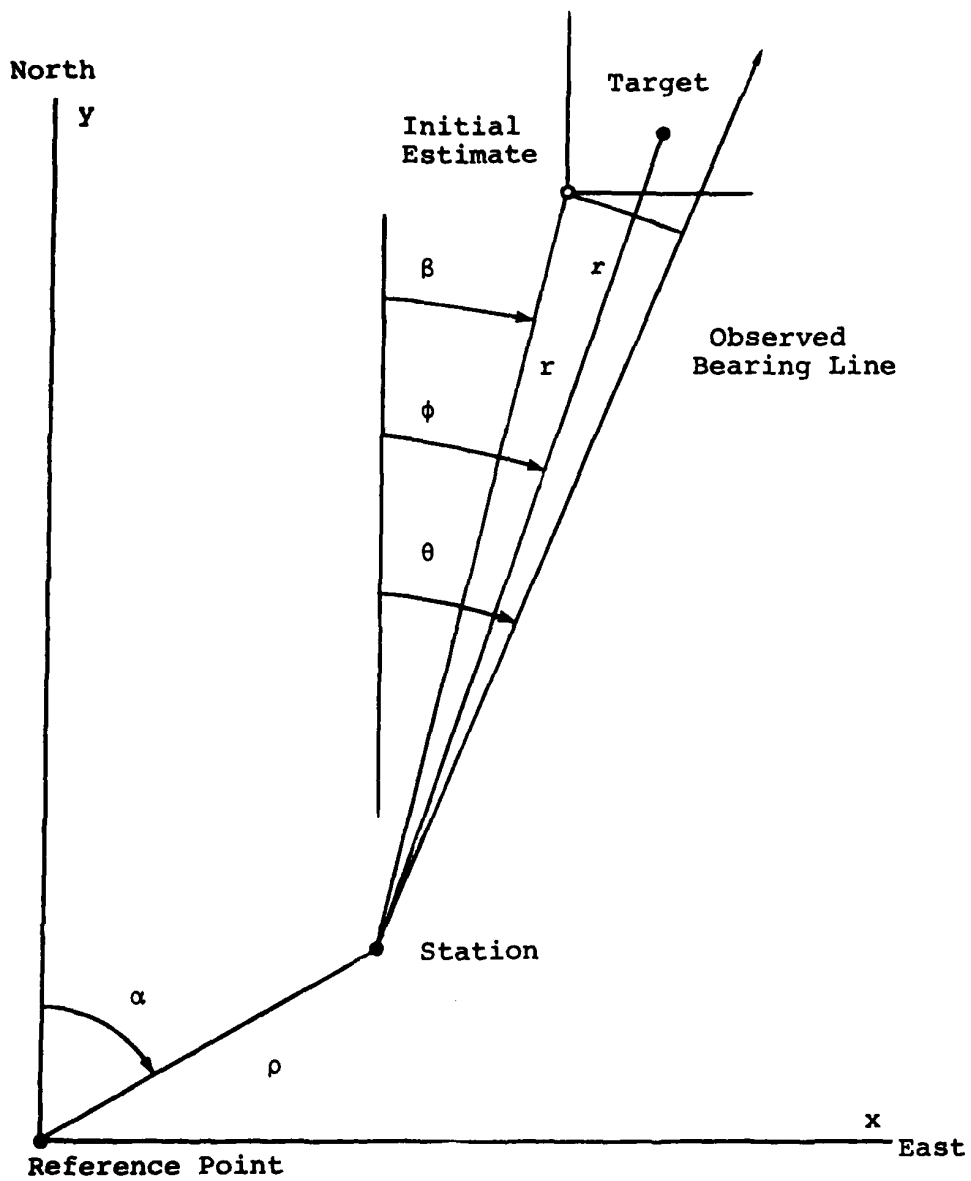
33

Figure 3. The coordinate geometry. The coordinates of the initial
estimate are (x*,y*). In the development, the reference
point is at the initial estimate.

34

also independent. And, consequently, the joint distribution of the U_i is determined.

To estimate x and y, maximum likelihood estimates are used here. The likelihood for a sample θ_1, θ_2, ..., θ_n from n stations is

$$L(\theta_1, \theta_2, \ldots, \theta_n) = \prod_1^n \frac{1}{\sqrt{2\pi}\ e_i} \quad \exp\ -\frac{1}{2} \sum_1^n (\theta_i - \phi_i)^2 / e_i^2$$

and the likelihood for a cooresponding sample u_1, u_2, ..., u_n is

$$L(u_1, u_2, \ldots, u_n) = \prod_1^n \frac{1}{\sqrt{2\pi}\ \sigma_i} \quad \exp\ -\frac{1}{2} \sum_1^n u_i^2 / \sigma_i^2$$

with $\sigma_i = r_i e_i$ where e_i is the standard deviation of θ_i.

By definition, the maximum likelihood estimates for x and y are the estimates which make $L(u_1, u_2, \ldots, u_n)$ a maximum. In this case, making $L(u_1, u_2, \ldots, u_n)$ a maximum is equivalent to making $\sum_1^n (u_i^2 / \sigma_i^2)$ a minimum. So, to find the maximum likelihood estimates $\hat{x}$ and $\hat{y}$, solve the following two equations for $\hat{x}$ and $\hat{y}$:

$$\frac{\partial(\ln L)}{\partial x}\Big| = 0 \quad \text{and} \quad \frac{\partial(\ln L)}{\partial y}\Big| = 0$$
$$x = \hat{x} \qquad\qquad y = \hat{y}$$

With $u_i = w_i - x \cos \beta_i + y \sin \beta_i$, the equations can be written as follows:

$$\sum_1^n (w_i - \hat{x} \cos \beta_i + \hat{y} \sin \beta_i)(\cos \beta_i)/\sigma_i^2 = 0$$

and

$$\sum_1^n (w_i - \hat{x} \cos \beta_i + \hat{y} \sin \beta_i)(\sin \beta_i)/\sigma_i^2 = 0 \ .$$

35

In terms of the following quantities:

$$A = \Sigma(\cos^2 \beta_i)/\sigma_i^2 \, , \qquad\qquad B = \Sigma(\sin \beta_i \cos \beta_i)/\sigma_i^2 \, ,$$

$$C = \Sigma(\sin^2 \beta_i)/\sigma_i^2 \, , \qquad\qquad D = \Sigma(w_i \cos \beta_i)/\sigma_i^2 \, ,$$

$$E = \Sigma(w_i \sin \beta_i)/\sigma_i^2 \, ,$$

the equations become:

$$A\hat{x} - B\hat{y} = D$$

$$B\hat{x} - C\hat{y} = E$$

The solutions are:

(1) $$\hat{x} = (BE - CD)/(B^2 - AC)$$

and

(2) $$\hat{y} = (AE - BD)/(B^2 - AC)$$

A confidence region can be constructed about an estimated position. In order to indicate how this can be done, a probability region about the true position will be considered first.

Both $\hat{x}$ and $\hat{y}$ are values of random variables. If a new set of bearings θ_1, θ_2, ..., θ_n is observed (for the same initial estimate and a fixed target), in general, a new pair of values $\hat{x}$ and $\hat{y}$ will be obtained.

If $\hat{X}$ and $\hat{Y}$ represent these random variables,

$$\hat{X} = \frac{1}{(B^2-AC)} \sum_{1}^{n} (W_i/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)$$

$$\hat{Y} = \frac{1}{(B^2-AC)} \sum_{1}^{n} (W_i/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)$$

where $W_i = r_i(\Theta_i - \beta_i)$.

Since $\hat{X}$ and $\hat{Y}$ are a linear combination of the n normal random variables W_1, W_2, ..., W_n, or equivalently of the n normal random variables Θ_1, Θ_2, ..., Θ_n, they have a joint normal distribution. Since $E(W_i) = r_i(\phi_i - \beta_i)$, if $\beta_i = \phi_i$ for i = 1, 2, ..., n, that is, if the initial estimate of the target's position is at the target's position, $E(W_1) = 0$ for i = 1, 2, ..., n. In this case $E(\hat{X}) = 0$ and $E(\hat{Y}) = 0$ and the joint normal distribution is centered on the object's position. To the degree of the approximations that have been made, this is also true if the initial estimate is not at the target's position.

A region of minimum area for a given probability of containment of an estimated position can be determined. The region is bounded by an ellipse which is centered on the object's position and whose axes lie along the axes of an x'y'-coordinate system that is obtained by rotating the xy-coordinate system that is centered on the object's position through an angle γ. In this system, $\sigma_{\hat{x}',\hat{y}'}$ is 0, that is $\hat{x}'$ and $\hat{y}'$ are independent normal random variables. The two coordinate systems are illustrated in

Figure 4. The coordinates of a point in the two systems are related by

$$x' = x \cos \gamma - y \sin \gamma$$

$$y' = x \sin \gamma + y \cos \gamma$$

These relations imply:

(3) $\quad \sigma_{\hat{x}'}^2 = \sigma_{\hat{x}}^2 \cos^2 \gamma - 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma_{\hat{y}}^2 \sin^2 \gamma$,

(4) $\quad \sigma_{\hat{y}'}^2 = \sigma_{\hat{x}}^2 \sin^2 \gamma + 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma_{\hat{y}}^2 \cos^2 \gamma$

and

(5) $\quad \sigma_{\hat{x}'\hat{y}'}^2 = (\sigma_{\hat{x}}^2 - \sigma_{\hat{y}}^2) \sin \gamma \cos \gamma + \sigma_{\hat{x}\hat{y}} (\cos^2 \gamma - \sin^2 \gamma)$

where γ, the angle of rotation of the coordinate axes, is positive in the clockwise direction. And $\sigma_{\hat{x}'\hat{y}'} = 0$ implies

$$\tan 2\gamma = \frac{2\sigma_{\hat{x}\hat{y}}}{\sigma_{\hat{y}}^2 - \sigma_{\hat{x}}^2}$$

With the initial estimate of the target's position at the target's position $E(W_i) = 0$ and therefore, $Var(W_i) = \sigma_i^2$ for $i = 1, 2, \ldots, n$. In this case

$$\sigma_{\hat{x}}^2 = \frac{1}{(B^2 - AC)^2} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)^2 ,$$
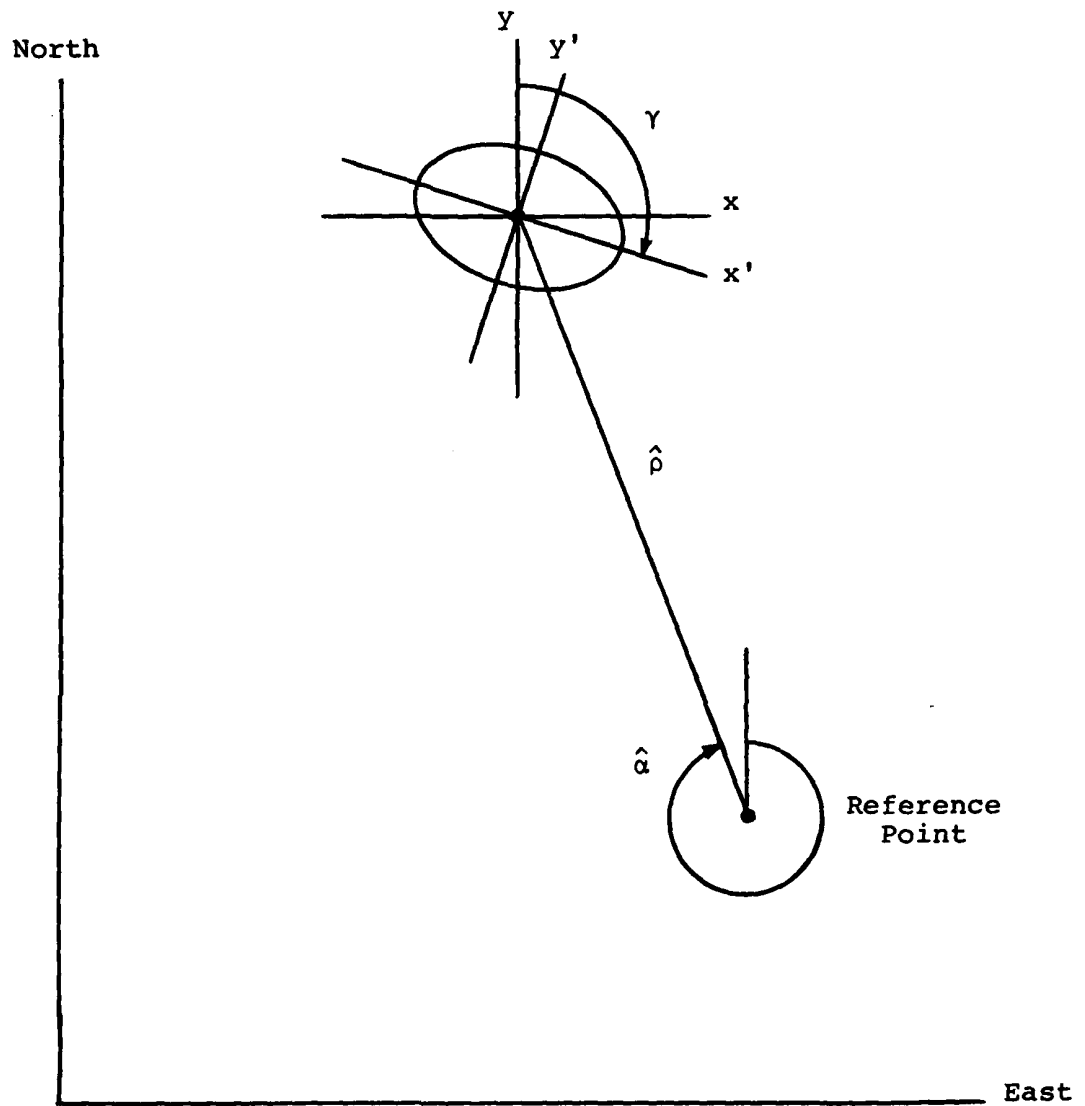
38

Figure 4. An elliptical confidence region and the primed coordinate system in which the covariance $\sigma_{\hat{x}',\hat{y}'}$ is zero. The center of the ellipse and the origin of the coordinate systems are at the target's estimated position. The estimated bearing $\hat{\alpha}$ and estimated range $\hat{\rho}$ are indicated for a reference position.

$$\sigma_{\hat{y}}^2 = \frac{1}{(B^2-AC)^2} \sum_1^n (1/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)^2$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{1}{(B^2-AC)} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)(A \sin \beta_i - B \cos \beta_i).$$

Using the definition for A, B and C, the above become

(6)
$$\sigma_{\hat{x}}^2 = \frac{C}{(AC-B^2)} ,$$

(7)
$$\sigma_{\hat{y}}^2 = \frac{A}{(AC-B^2)} ,$$

and

(8)
$$\sigma_{\hat{x}\hat{y}} = \frac{B}{(AC-B^2)} .$$

So tan 2 = 2B/(A-C) for $\beta_i = \phi_i$, i = 1, 2, ..., n.

With the target's position known and, consequently, ϕ_i known for i = 1, 2, ..., n, the above expressions for $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$, $\sigma_{\hat{x}\hat{y}}$ and γ can be used, since the initial estimate of the target's position can be taken as the target's position.

With values for $\sigma_{\hat{x}}$, $\sigma_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ, values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be found by using equations (3) and (4). The probability that an estimated position will be within an ellipse of semiaxes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$, which is centered on the target's position is

$1 -\exp(-k^2/2)$. This result can be found by integrating the bi-
variate normal density over the ellipse. And the area of the
ellipse is $k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$.

Given estimates $\hat{x}$ and $\hat{y}$ found by using Equations (1) and (2),
the ellipse with semi-axes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ in a coordinate system
that is centered on the point $(\hat{x},\hat{y})$ and has been rotated through
an angle γ is a $1 - \exp(-k^2/2)$ confidence region. This follows,
since, to the degree of the approximations involved, the bivar-
iate normal distribution of X and Y is centered on the target's
position. The confidence ellipse is defined if $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$ and $\sigma_{\hat{x}\hat{y}}$
can be found, that is if the elements of the covariance matrix
can be found. To the degree of the approximations involved,
this can be done as follows: First, assume the initial estimate
of the target's position is at the target's position. Then,
values for $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$, $\sigma_{\hat{x}\hat{y}}$ and γ can be determined by using Equations
(6), (7) and (8). These values can then be used to determine
$\sigma_{\hat{x}'}^2$, $\sigma_{\hat{y}'}^2$, and $\sigma_{\hat{x}'\hat{y}'}$ by using Equations (3), (4) and (5). Now,
with a value for k, a confidence region can be constructed. To
the degree of the approximations involved, the shape of the con-
fidence region is independent of both the target's position and
of the initial estimate of the target's position.

For the case where bearings are taken from the target on two
or more stations, θ_i is the reciprocal of the bearing taken from
the target.

A discussion of the theory of bearings only position
estimation procedures for situations similar to the one considered
here is given in Reference 1. Reference 2 gives an essentially

equivalent bearings only procedure. It also gives a range only
procedure, a range and bearing procedure and HP-9830A programs
with which to implement the procedures. Using the fix determined
by two lines of bearing as the initial estimate was suggested by
this reference.

The following equations are evaluated in the program to deter-
mine the coordinates of the initial estimate:

$$x^* \sin(\theta_2 - \theta_1) = [\rho_1 \sin(\alpha_1 - \theta_1)] \sin \theta_2$$

$$- [\rho_2 \sin(\alpha_2 - \theta_2)] \sin \theta_1$$

and

$$y^* \sin(\theta_2 - \theta_1) = [\rho_1 \sin(\alpha_1 - \theta_1)] \cos \theta_2$$

$$- [\rho_2 \sin(\alpha_2 - \theta_2)] \cos \theta_1 .$$

Reference 3 describes a TI-59 program that is based on an equi-
valent procedure. The program allows a user to either input
the coordinates of the initial estimate or determine them in
the manner described here.

Appendix 2. HP-41CV Program Labels

The global labels in the HP-41CV program that are assigned
to the keys Σ+, 1/x, √x, LOG and LN give these keys a mnemonic
character. For example, with the calculator in USER mode, if
LOG is pressed and held, SIZ will be displayed and then after
a delay, NULL.

If a global label is replaced by a local label, this mnemonic
character will be lost. However, automatic key assignment will
be gained if the key corresponding to the local label has not
been previously assigned. If this is the case and if there is
a second program in program memory that uses the same local label,
then it will be automatically assigned when the calculator is
positioned in program memory at that program.

Global labels in the HP-41CV program can be replaced with
local labels. If this is done as described below, the user in-
structions will still be applicable. First, make the following
replacements:

Line 329: LBL "PRB" → LBL E.   Line 311: LBL "SIZ" → LBL D.

Line 186: LBL "EST" → LBL B.   Line 008: LBL "CON" → LBL C.

Line 047: GTO "CON" → GTO C.

Next, after line 001: LBL "TPE", insert LBL A so that it
becomes line 002. Finally, if there is a key assignment for a
key in the following list: Σ+, 1/x, √x, LOG and LN, remove it.
Now, with the calculator in USER mode, if LOG is pressed and held,
XEQ D will be displayed and then, after a delay, NULL. The key's
mnemonic character has been lost, but if there is a second pro-
gram in program memory with the local label D, LOG will be auto-
matically assigned to D when the calculator is positioned in pro-
gram memory at that program.

REFERENCES

1. Daniels, H. E., "The Theory of Position Finding," _J. Royal Stat. Soc. (B)_, Vol. 13, 1951, pp. 186-207.

2. Thompson, K. P. and Kullback, J. H., "Position-Fixing and Position-Predicting Programs for the Hewlett-Packard Model 9830A Programmable Calculator," NRL Memorandum Report 3265, Naval Research Laboratory, Washington, DC , April 1976.

3. Forrest, R. N., "A Procedure for Estimating an Object's Position Based on Two or More Bearings with a Program for a TI-59 Calculator," NPS55-77-34 (Revised), Naval Post-graduate School, Monterey, CA., 93940, September 1977 (Revised August 1978).

INITIAL DISTRIBUTION LIST

COPIES

Dean of Research Code 012 Naval Postgraduate School Monterey, California 93940	1
Defense Technical Information Center Cameron Station Alexandria, Virginia 22314	2
Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
Office of Naval Research Attn: Code ONR-230 800 North Quincy Street Arlington, Virginia 22217	3
Commander Submarine Development Squadron 12 Naval Submarine Base, New London Groton, Connecticut 06340	1
Commander Surface Warfare Development Group Naval Amphibious Base, Little Creek Norfolk, Virginia 23521	1
Director Strategic Systems Projects Office Attn: Code SP2024, Code 20231 1931 Jefferson Davis Highway Arlington, Virginia 20376	2
Navy Tactical Support Activity Attn: M. E. Lannon P. O. Box 1042 Silver Spring, Maryland 20910	1
Naval Air Development Center Johnsville, Pennsylvania 18974	1
Naval Surface Weapons Center White Oak Silver Spring, Maryland 20910	1
Naval Underwater Systems Center Newport, Rhode Island 02840	1
Naval Underwater Systems Center New London, Connecticut 06320	1

Naval Ocean Systems Center   1
San Diego, California   92132

Naval Intelligence Support Center   2
Attn: Code NISC 20, Library
4301 Suitland Road
Washington, DC   20390

Naval Electronic Systems Command   1
2511 Jefferson Davis Highway
Arlington, Virginia   20360

Naval Air Systems Command   1
Code 370
Washington, DC   20361

Antisubmarine Warfare Systems Project Office   1
Department of the Navy
Washington, DC   20360

Center for Naval Analysis   1
1401 Wilson Boulevard
Arlington, Virginia   22209

Code 55Fo   200
Naval Postgraduate School
Monterey, California 93940