AERODYNAMIC CALCULATION METHODS for PROGRAMMABLE CALCULATORS & PERSONAL COMPUTERS

— with programs for the TI-59 —

PAK #3
BASIC SUBSONIC AERODYNAMICS

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AEROCAL

Preface

AEROCAL programs are intended to serve both students and practicing aerodynamicists. For students, they can serve an important role in supplementing theoretical analysis with the actual numerical results so important in developing engineering skills. In aerodynamics, it has been difficult for students to solve meaningful illustrative problems, and this difficulty can now be eliminated by using the new personal computing machines -- either programmable calculators or microcomputers. I have found that the results of numerical calculations inevitably provide a few surprises, which force the analyst to reexamine the theory, leading to a much deeper understanding. AEROCAL programs can thus be used to prevent the calculation from becoming an end in itself. Instead, efforts can be concentrated on the actual aerodynamic problems, with required calculations assuming their proper supporting role. Thus, the availability of the personal computing machines allows the student to gain an appreciation of the role of computational aerodynamic simulations, while developing an engineering attitude.

The second purpose of the work is to provide the practicing aerodynamicist with a readily accessible collection of algorithms designed for use on this class of machine. The availability of such a set of routines will eliminate the most tedious aspects of the software development process so that the code development time can be used to implement the user's unique requirements rather than wasting time creating the basic building blocks.

The material selected for inclusion is, of course, not intended to replace the large computational aerodynamics programs. Instead, it allows students to become familiar with an important part of the set of standard aerodynamic methods representative of those required in aerodynamics. To the experienced user, these methods should be extremely useful, providing results which are more than adequate for a variety of jobs.

The material is organized in workbook fashion, with each program being essentially independent of the others. An example of the style that we intend to follow is found in the IBM SSP or other software package user's manuals. The addition of some examples for each program allows the user to check that the program is properly executing on his own machine.

The choice of the TI59 format for the programs is one of convenience only. Program instructions are similar for other calculators and an Appendix is included to describe the listing nomenclature. Using this information, conversion to other instruction sets should be relatively simple. Microcomputers will typically have more advanced instruction sets, such as BASIC. The information provided in the method description is easily used to write a set of BASIC instructions.

The author acknowledges the contributions of the many aerodynamicists and research scientists who have developed the basic material, which forms the basis for these software paks and with whom he has held discussions on the relative merits of particular methods for performing various aerodynamics calculations.

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3.0 INTRODUCTION

The prediction of pressure distributions and forces on aerodynamic shapes at subsonic speeds provides the basis of most aerodynamic design and analysis. Programmable calculators and microcomputers cannot as yet compute the flowfield over entire configurations. But methods which compute the flows over components can be handled on small machines and provide useful results.

Two methods which give exact results for idealized shapes have been included. The first is the classical Joukowski airfoil solution, which demonstrates the use of conformal transformation and can be used to study how pressure distributions change with the variation of geometric parameters. It is also an excellent shape to use as a check of the solutions produced by more approximate methods. Some useful airfoil relations are also summarized in this section. The second exact solution is given for an ellipsoid of revolution at angle-of-attack. This solution can provide information on the typical magnitude of pressures and crossflow angles on fuselage-like shapes.

For arbitrary airfoil analysis, Weber's 2nd order analysis method is included, while design load distributions can be obtained with the camber line design method described in Section 3.4. These methods provide a solid basis for 2-D incompressible, inviscid airfoil analysis and design.

For wings, a lifting line program is described which can be used to determine lift, lift curve slope, induced drag and spanload for arbitrary unswept wings. Finally, a Trefftz plane analysis of the spanload is presented which allows the user to determine the spanload efficiency 'e'.

The methods can be readily extended using the references and a microcomputer to treat other situations. Obvious simple extensions include compressibility effects in Weber's method and the addition of sweep effects in the lifting line theory method. Thus, the methods presented here can serve as a foundation for the use of other methods and to obtain useful information in a number of cases typical of aerodynamic design and analysis.

The methods are presented in a standard format with the following information:

- o Title
- o Description of what the method does
- o References
- o Detailed outline of the method and listing of the equations required
- o User instructions
- o Sample case
- o Program description
- o Program listing.

The programs are written in the most direct sequence of instructions possible in order to make the study of the programs as simple as possible. This allows the user to incorporate modifications to the programs or convert them to other systems without difficulty. The use of the TI59 instruction set is purely a convenient selection. These routines will work on a number of other calculators, as well as the emerging class of microcomputers. For those readers not familiar with the details of the TI59 instruction set, a description is included in Appendix A. This will allow the non-TI59 user to convert the codes to his own instruction set with ease.

The routines often make use of the printer. The author has found the printer to be much more valuable in program development than in program execution. Nevertheless, several programs do provide printed results. A description of the printed output is included below the user instructions for each program.

3.1 JOUKOWSKI AIRFOIL

This program computes the Joukowski airfoil ordinates and related surface pressure and velocity. The method employed is the one presented by E. L. Houghton and A. E. Brock, Aerodynamics for Engineering Students, Edward Arnold, London, 1960, pp. 281-298. This method allows for a slightly more general form of the airfoil and pressure distribution. Once the foil and pressure are known, the force and moment integrations are carried out, together with the arc length calculation. The arc length is mainly useful in making boundary layer calculations.

METHOD

The Joukowski airfoil is one of the shapes resulting from the Joukowski transformation:

$$\zeta = Z + \frac{b^2}{Z^2}$$

where $\zeta = \xi + i\eta$ represents the physical plane, and $Z = x + iy$ corresponds to the transformed plane.

The idea is to relate the flow about a circular cylinder of radius "a" in the transformed plane to the related shape in the physical plane. The value of a (compared to b) and the displacement of the cylinder from the origin determine the shape in the physical plane. Basically, displacement of the cylinder along the x axis controls the thickness, while displacement above the x axis controls camber.

Therefore, to start the computation, select the geometric parameters:

e: where the thickness $t/c \approx 1.3e$

β: where the max camber is about $\beta/2$ (β is in radians)

b: where the chord in the physical plane is about 4b

a: for the standard Joukowski foil $a = b\,(1 + e)$

and the flow conditions:

α: the angle-of-attack

K: the fraction of the Kutta condition achieved (typically from .9 to 1.0).

Once these values are specified, the calculation is made at each point in the plane, moving sequentially around the circular cylinder.

For each θ, the following simple algebraic steps are carried out:

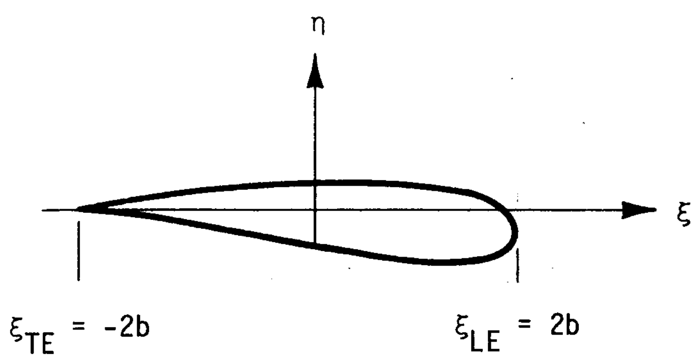
Compute: $x = a \cos \theta + be$

$$y = a \sin \theta + \beta \cdot b\,(1 + e)$$

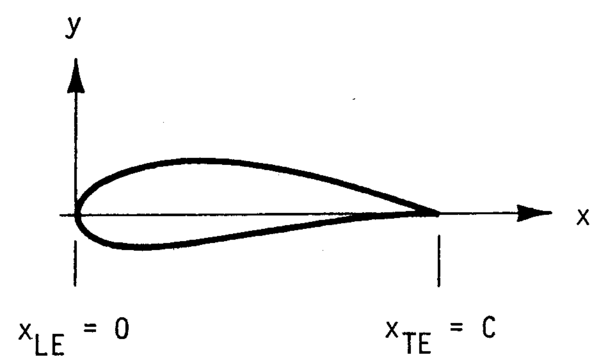
$$\xi = x \left\{ 1 + \frac{b^2}{x^2 + y^2} \right\}, \quad \eta = y \left\{ 1 - \frac{b^2}{x^2 + y^2} \right\}$$

ξ and η are the coordinates of the Joukowski airfoil.

The coordinates that evolve naturally in the physical plane are not the usual ones associated with airfoils. The ξ, η coordinate system in the following sketch can be compared with the standard airfoil coordinate system:



Joukowski Coordinates



Standard Coordinates

Once the airfoil ordinates are known, the surface velocity, qe, and pressure distribution, C_p, are:

$$A = 1 - b^2 \frac{(x^2 - y^2)}{[x^2 + y^2]^2}$$

$$B = \frac{2\,b^2\,xy}{(x^2 + y^2)^2}$$

$$\frac{qe}{U_\infty} = \frac{2\left\{\sin(\theta + \alpha) + K \sin(\alpha + \beta)\right\}}{\left[A^2 + B^2\right]^{1/2}}$$

$$C_p = 1 - \left(\frac{qe}{U_\infty}\right)^2$$

Note that if $A^2 + B^2 = 0$, the mapping is at a singular point and the Kutta condition is invoked to automatically set $qe/U_\infty = 0$.

SOME TYPICAL SHAPES

A number of special shapes result from the appropriate combinations of geometric parameters:

i) Flat plate: $e = 0$, $a = b$, $\beta = 0$

ii) Circular arc airfoil: $e = 0$, $b = a \cos \beta$

iii) Ellipse: $e = 0$, $\beta = 0$, $b < a$

where the fineness ratio (chord/max thickness) is $(a^2 + b^2)/(a^2 - b^2)$.

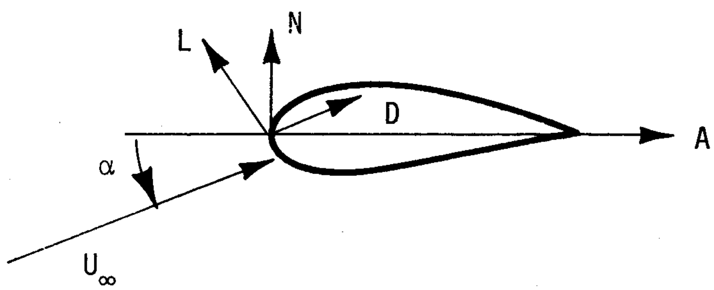
Note also that the trailing edge of the Joukowski airfoil is cusped. Although the transformation cannot be manipulated to produce a finite trailing edge angle, the parameters can be adjusted to allow a slightly rounded trailing edge. This is done by shifting the cylinder less than the full amount required to place the singular point of the mapping exactly on the cylinder; i.e., a is larger than it should be:

$$a = b\,(1 + Qe)$$

where Q is greater than one (it is two in the example by Houghton and Brock). This change also shifts the maximum thickness aft on the foil.

FORCE AND MOMENT CALCULATIONS

Once the pressures are known, the forces can be computed. Generally, the calculations are made in the airfoil coordinate system (body axis) and then rotated to the aerodynamic system (wind axis).



such that:

$$C_L = C_N \cos \alpha - C_A \sin \alpha.$$

$$C_D = C_N \sin \alpha + C_A \cos \alpha.$$

Conversely, it is occasionally useful to use the inverse relations:

$$C_N = C_L \cos \alpha + C_D \sin \alpha.$$

$$C_A = -C_L \sin \alpha + C_D \cos \alpha.$$

For the 2-D potential flows in aerodynamics, $C_D = 0$, $C_A = -C_L \sin \alpha$ and $C_L = C_N/\cos \alpha$.

In order to obtain C_N and C_m, the pressures can be integrated by the trapezoidal rule over the surface.

As a check, the analytic result for C_L on a Joukowski foil is given by:

$$C_L = 2\pi K \left(\frac{4a}{c}\right) \sin (\alpha + \beta)$$

where c is the airfoil chord and must be obtained from the ordinate calculation.

The formulas for the force and moment summation (together with the formula for the arc length over the surface are given by:

Normal Force:

$$\frac{N}{q} = \sum \left\{\frac{Cp_i + Cp_{i-1}}{2}\right\} \left\{\xi_i - \xi_{i-1}\right\}$$

Moment:

$$\frac{M}{q} = \sum \left\{\frac{Cp_i + Cp_{i-1}}{2}\right\} \left\{\Delta\xi \left(\xi_{i-1} + \frac{\Delta\xi}{2}\right) + \Delta\eta \left(\eta_{i-1} + \frac{\Delta\eta}{2}\right)\right\}$$

Arc Length:

$$S = \sum \sqrt{\Delta\xi^2 + \Delta\eta^2}$$

Here, N/q must be divided by c and M/q should be divided by c^2 to obtain C_N and C_m, respectively.

The reference location for the origin of the moments is the origin of the ξ, η coordinate system. The moment about any other point can be obtained as follows:

i) Convert to the standard airfoil coordinate system and define the moment origin as:

$$\frac{x_R}{c} = \frac{\xi_{LE} - \xi_R}{\xi_{LE} - \xi_{TE}}, \qquad \frac{y_r}{c} = \frac{\eta_R}{\xi_{LE} - \xi_{TE}}$$

where

$$\xi_R = \eta_R = 0 \rightarrow \frac{x_R}{c} = \frac{\xi_{LE}}{\xi_{LE} - \xi_{TE}} , \quad \frac{y_R}{c} = 0.$$

ii) Define the coordinates about which the moment is desired:

$$\left(\frac{x_Q}{c} , \frac{y_Q}{c}\right) .$$

iii) The moment about x_Q, y_Q is then given by:

$$Cm_Q = Cm_R + \left(\frac{x_Q}{c} - \frac{x_R}{c}\right) C_N - \left(\frac{Y_Q}{c} - \frac{Y_R}{c}\right) C_A.$$

For the Joukowski airfoil $Y_R = 0$, $C_A = -\tan \alpha \; C_N$ and the moment transfer reduces to:

$$Cm_Q = Cm_R + \left[\left(\frac{x_Q}{c} - \frac{x_R}{c}\right) + \tan \alpha \cdot \frac{y_Q}{c}\right] C_N .$$

Two other quantities are also of interest; the aerodynamic center and the center of pressure.

The aerodynamic center (ac) is the point about which the moment is constant (i.e., it does not change with angle-of-attack). The location of x_{ac} is usually approximated as:

$$\frac{x_{ac}}{c} = \frac{x_R}{c} - \frac{dCm_R}{dC_L} .$$

The moment about the aerodynamic center can be estimated by assuming that it is approximately equal to the value of Cm_R at $C_L = 0$:

$$Cm_{ac} \approx Cm_R \; (C_L = 0) = Cm_o.$$

The center of pressure (cp) is the point on the airfoil about which the moment is zero. The position of the cp is usually computed approximately as:

$$K_{cp} \cong \frac{x_{ac}}{c} - \frac{Cm_{ac}}{C_L} .$$

USER INSTRUCTIONS -- PROGRAM 3.1

STEP	ENTER	PRESS	DISPLAY
1. x-offset, e $\left[e \approx .77 \left(\frac{t}{c}\right)\right]$	e	A'	-189.
2. Y-offset, β°* $\left[\beta^\circ \approx \left(\frac{360}{\pi}\right)\left(\frac{Z_{CAMMAX}}{c}\right)\right]$ Default is β = 0°.	β (Degrees)	B'	β
3. Transform constant, b [c ∿ 4b] Default is b = 1.	b	C'	a
4. Radius, a {input only if a≠b(1+e)}	a	D'	a
5. Fraction of Kutta condition, K (default is K=1)	K	E'	K
6. Starting point for solution [default is θ_i=-180°, the lower surface trailing edge].	θ_i	A	θ_i
7. Step size for solution [default Δθ = 9°].	Δθ	B	$\theta_i - \Delta\theta$
8. Angle-of-attack, α	α	C	
9. Solution at θ_i	-	D	Cp
		RCL 24	θ
Repeat Step 9 until all θ's are computed. Note that solution moves forward on lower surface, around l.e. and back upper surface to the trailing edge.		RCL 18	ξ
		RCL 19	η
		RCL 35	S
		RCL 34	I_{C_L}
I_{C_L} and I_{C_m} are the force summations and are usually only of interest when the calculation is completed all the way around.		RCL 36	I_{C_m}

* All angles are entered in degrees.

NOTES: (Continued on next page.)

USER INSTRUCTIONS -- PROGRAM 3.1 (Continued)

NOTES:
1. The 1st 5 steps have to be completed in order for each case.
2. Steps 6 and 7 can be reset to study local effects.
3. The listing assumes that the printer will be used. If it is not used, Steps 353, 354 and 355 should be deleted.
4. If the printer is used, unlabeled output is generated in the following order: each input value is echoed, followed by θ, ξ, η, C_p, s for each step. Finally, if the printer is used, Step 9 is required only once and the program automatically steps around the foil.

SAMPLE CASE: e = .1, b = 1., a = 1.1, K = 1.

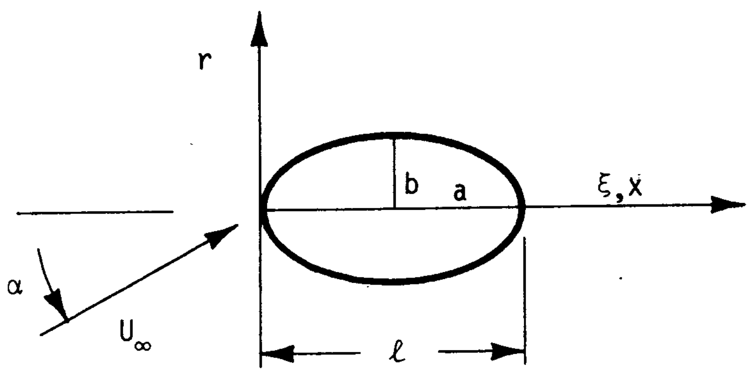
$\theta_i = -180°$, $\Delta\theta_i = 9°$, $\alpha = 6°$

	$\beta = 0.0$				$\beta = 4.6°$		
$\theta°$	ξ	η	C_p		ξ	η	C_p
-180.	-2.000	0.0000	1.0000		-1.992	.0007	.1801
-162.	-1.882	-.0036	.1927		-1.933	.0109	.2845
-144.	-1.548	-.0261	.1729		-1.634	.0384	.3620
-117.	- .756	-.1051	.1056		- .818	.0422	.3562
- 90.	.182	-.1984	.0353		.197	-.0328	.3145
- 63.	1.054	-.2375	.0479		1.119	-.1194	.2869
- 36.	1.698	-.1841	.3321		1.756	-.1260	.4995
- 18.	1.948	-.1021	.8649		1.979	-.0689	.9890
- 9.	2.012	-.0524	.9273		2.025	-.0246	.3068
0.	2.033	0.0000	- .8724		2.029	.0273	-2.0112
9.	2.012	0.0524	-2.5229		1.991	.0839	-2.4330
18.	1.948	.1021	-2.3040		1.912	.1422	-2.1898
36.	1.698	.1841	-1.5561		1.641	.2514	-1.7670
63.	1.054	.2375	- .9037		.999	.3565	-1.3359
90.	.182	.1984	- .4711		.170	.3527	- .9282
117.	- .756	.1051	- .1576		- .706	.2472	- .5205
144.	-1.548	.0261	+ .0518		-1.469	.1035	- .1592
162.	-1.882	.0036	+ .1371		-1.8234	.0312	+ .0340
180.	-2.000	0.0000	1.0000		-1.992	.0007	+ .1801

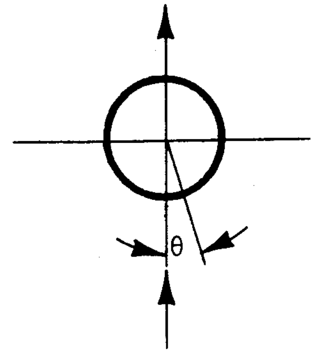
$\beta = 0.0$	$\beta = 4.6°$
c = 4.033, s = 8.2260	c = 4.021, s = 8.2397
$\frac{N}{q} = 2.864$, $\frac{M}{q} = 2.859$	$\frac{N}{q} = 5.0399$, $\frac{M}{q} = 3.032$
$C_N = .710$, $C_L = .714$	$C_N = 1.253$, $C_L = 1.260$
$C_m = .1758$	$C_m = .1875$

3.2 ELLIPSOID OF REVOLUTION

This program computes the surface flow over an ellipsoid of revolution at angle-of-attack. The notation for this class of bodies is shown in the sketch. They are more formally known as prolate spheroids. This particular formula is from the article by Cebeci, T., Kattab, A. H. and Stewartson, K., "On Nose Separation," Journal of Fluid Mechanics, Vol. 97, Pt. 3, pp. 435-454, 1980.



GEOMETRY NOTATION



$U_\infty \sin\alpha$

METHOD OF CALCULATION

i) Specify the fineness ratio, a/b, and the angle-of-attack α.

ii) Define $t = \frac{b}{a}$ and compute V_0 (t) and V_{90} (t):

$$V_0(t) = \frac{(1 - t^2)^{3/2}}{(1 - t^2)^{\frac{1}{2}} - \frac{t^2}{2} \ell n \left[\frac{1 + (1 - t^2)^{1/2}}{1 - (1 - t^2)^{1/2}}\right]}$$

and

$$V_{90}(t) = \frac{2 V_0(t)}{2 V_0(t) - 1} .$$

iii) Given (x/ℓ), define $\xi = 2\left(\frac{x}{\ell}\right) - 1$ and compute:

$$\frac{r}{\ell} = \frac{t}{2}\sqrt{1 - \xi^2}$$

and

$$\cos\hat{\beta} = \frac{(1 - \xi^2)^{1/2}}{\left[1 - \xi^2 (1 - t^2)\right]^{1/2}}$$

where if $\xi > 0$, $\hat{\beta}$ is negative.

iv) Given θ, compute the surface velocity components:

$$\frac{U_e}{U_\infty} = V_o(t)\cos\alpha \cdot \cos\hat{\beta} - V_{90}(t)\sin\alpha\sin\hat{\beta}\cos\theta$$

$$\frac{W_e}{U_\infty} = V_{90}(t)\sin\alpha\sin\theta$$

v) Finally, compute the pressure coefficient

$$C_p = 1 - \left[\left(\frac{U_e}{U_\infty}\right)^2 + \left(\frac{W_e}{U_\infty}\right)^2\right]$$

and cross flow angle relative to the x axis of:

$$\beta = \tan^{-1}\left(\frac{W_e}{U_e}\right)$$

USER INSTRUCTIONS -- PROGRAM 3.2

STEP	ENTER	PRESS	DISPLAY
1. Input the fineness ratio	$\frac{a}{b}$	A	V_{90} (t)
2. Input angle-of-attack in degrees	α	B	α
3. Input fuselage station	$\frac{x}{\ell}$	C	$\hat{\beta}$
4. Input circumferential station	θ	D	β
		RCL 13	U_e/U_∞
		RCL 14	W_e/U_∞
		RCL 15	C_p

NOTE: i) For a constant X/ℓ, repeat Step 4. for each θ.
ii) For a new X/ℓ, repeat Step 3. and 4.

If the printer is employed, unlabeled output is generated in the following order: input from Steps 1-4 is echoed, followed by U_e/U_∞, W_e/U_∞, C_p and β.

* When $U_e = 0$, β is undefined without additional analysis and the display flashes. The rest of the program output is still valid.

SAMPLE CASE: $\frac{a}{b} = 4$, $\alpha = 40°$

X/ℓ	C_p (θ = 0°)	C_p (θ = 90°)	C_p (θ = 180°)
0.000	-.4291	- .4291	- .4291
0.010	.8355	- .7019	-1.1033
0.015	.9336	- .7731	-1.0473
0.040	.9880	- .9397	-0.7417
0.250	.5785	-1.1015	0.0183
0.500	.3136	-1.1155	0.3136
0.750	.0183	-1.1015	0.5785
0.970	-.8508	- .8946	1.0000

3.3 AIRFOIL ANALYSIS USING WEBER'S 2ND ORDER METHOD

This program computes the pressure distribution over arbitrary airfoil shapes in incompressible inviscid flow. The method employed was presented by J. Weber in two British reports, "The Calculation of the Pressure Distribution Over the Surface of Two-Dimensional and Swept Wings with Symmetrical Aerofoil Sections," British ARC R&M 2918, July 1953 and "The Calculation Of the Pressure Distribution On the Surface of Thick Cambered Wings and the Design of Wings with Given Pressure Distribution," ARC R&M 3026, June 1955. A related report which provides a great deal of information on this class of methods is by M. D. van Dyke, "Second-Order Subsonic Airfoil Theory Including Edge Effects," NACA R-1274, 1956. The theory underlying Weber's approach is described in detail in the text by E. L. Houghton and R. P. Boswell, Further Aerodynamics for Engineering Students, St. Martin's, New York, 1969, pp. 77-89. These references should be consulted for the theoretical background.

In the particular example program presented, the airfoil coordinate subroutine is for the NACA 4 digit airfoils. Different airfoils will require a new subroutine. The code has been designed so that the airfoil routine is independent of the method and using the code description presented, new routines can be easily generated.

Once the pressure distribution is determined, a second program can be used to integrate the pressure to obtain the lift and moment.

a) Pressure Calculation

At the points $X_\nu = \frac{1}{2}(1 + \cos\frac{\nu\pi}{N})$, the pressure is given by the formula

$$C_p = 1 - \frac{\left\{\cos\alpha\,[1 + R^1(X) \pm R^4(X)] \pm \sin\alpha \cdot \sqrt{(1-X)/X}\,[1 + R^3(X)]\right\}^2}{1 + [R^2(X) \pm R^5(X)]^2} .$$

Here, the plus sign is for the upper surface and the minus sign for the lower surface. N is the number of points for the calculation (actually N-1). N can be selected to be 8, 16 or 32, however, 16 is a reasonable upper limit (1 hour computing time on a TI59).

The R's are determined from the airfoil thickness and camber:

$$Z_t = \frac{1}{2}(Z_{up} - Z_{low})$$

$$Z_c = \frac{1}{2}(Z_{up} + Z_{low})$$

and the following summations:

$$R^1(X_\nu) = \sum_{\mu=1}^{N-1} S^1_{\mu\nu} \, Z_{t_\mu}$$

$$R^2(X_\nu) = \sum_{\mu=1}^{N-1} S^2_{\mu\nu} \, Z_{t_\mu} = \frac{dZ_t}{dx}$$

$$R^3(X_\nu) = \sum_{\mu=1}^{N-1} S^3_{\mu\nu} \, Z_{t_\mu} + S^3_{N\nu} \sqrt{\frac{\rho}{2c}}$$

$$R^4(X_\nu) = \sum_{\mu=1}^{N-1} S^4_{\mu\nu} \, Z_{c_\mu}$$

$$R^5(X_\nu) = \sum_{\mu=1}^{N-1} S^5_{\mu\nu} \, Z_{c_\mu} = \frac{dZ_c}{dx}$$

where ρ is the leading edge radius and the S functions are given below.

Note that if the airfoil is not cambered R^4 and R^5 are not needed, and that if the derivatives of the foil are known analytically, it may be better to supply them to the program directly.

The S functions depend on series analysis and, hence, numerous sines and cosines. Defining

$$\theta_\mu = \cos^{-1}(2X_\mu - 1) \text{ and } \theta_\nu = \cos^{-1}(2X_\nu - 1)$$

we also introduce $T_{\mu\nu} = (\cos\theta_\mu - \cos\theta_\nu)$.

The S's thus become:

$$S^1_{\mu\nu} = \frac{(-1)^{\mu-\nu} - 1}{N} \cdot \frac{2 \sin \theta_\mu}{T^2_{\mu\nu}} \qquad \mu \neq \nu$$

$$= \frac{N}{\sin \theta_\nu} \qquad \mu = \nu$$

$$S^2_{\mu\nu} = \frac{-2\,(-1)^{\mu-\nu} \sin \theta_\mu}{\sin \theta_\nu\, T_{\mu\nu}} \qquad \mu \neq \nu$$

$$= \frac{\cos \theta_\nu}{\sin^2 \theta_\nu} \qquad \mu = \nu$$

$$S^3_{\mu\nu} = S^1_{\mu\nu} + \frac{2}{N}\, \frac{1-(-1)^{\mu-\nu}}{\sin \theta_\mu} \cdot \frac{1}{T_{\mu\nu}} \qquad \mu \neq \nu$$

$$= S^1_{\nu\nu} \qquad \mu = \nu$$

$$S^4_{\mu\nu} = \frac{2[(-1)^{\mu-\nu} - 1]}{N \sin \theta_\nu} \cdot \frac{1 - \cos \theta_\mu \cos \theta_\nu}{T_{\mu\nu}{}^2} - \frac{2[(-1)^\mu - 1]}{N \sin \theta_\nu\, (1 - \cos \theta_\mu)} \qquad \mu \neq \nu$$

$$= \frac{N}{\sin \theta_\nu} - \frac{2[(-1)^\nu - 1]}{N \sin \theta_\nu} \cdot \frac{1}{(1 - \cos \theta_\nu)} \qquad \mu = \nu$$

$$S^5_{\mu\nu} = \frac{-2\ (-1)^{\mu-\nu}}{T_{\mu\nu}} \qquad \mu \neq \nu$$

$$= -S^2_{\nu\nu} \qquad \mu = \nu$$

A special formula is required for S^3 at $\mu = N$:

$$S^3_{N\nu} = \frac{[(-1)^{\nu} - 1]}{N} \cdot \frac{1}{1 + \cos\theta_\nu} \quad .$$

In computers with some storage space, the S functions can be computed once and then stored for the rest of the calculation. However, most programmable calculators do not have that much storage and the S functions must be continually recomputed. This is the step which uses much of the computing time.

b) Force and Moment Calculation

Once the pressures are calculated, a second program is used to perform the integration for C_L and C_m:

$$C_L = \int_0^1 \Delta C_p \ dX$$

and

$$C_m = - \int_0^1 \Delta C_p \ (X - X_{REF}) \ dX.$$

Using $X = \frac{1}{2}(1 + \cos\theta)$, these integrals become:

$$C_L = \int_0^\pi \frac{\Delta C_p}{2} \sin\theta d\theta$$

and

$$C_m = - \int_0^\pi \frac{\Delta C_p}{4} [\cos\theta - \cos\theta_{REF}] \sin\theta d\theta.$$

Note that the C_L integral is made assuming that the small angle approximations are used. Using the trapezoidal rule and assuming $\Delta C_p = 0$ at $X = 0, 1$ (where the information is not available anyway) and $\Delta\theta = \pi/N$, the summations are:

$$C_L = \frac{\pi}{N} \sum_{n=1}^{N-1} \left(\frac{\Delta C_p}{2}\right)_n \sin\theta_n$$

$$C_m = -\frac{\pi}{N} \sum_{n=1}^{N-1} \left(\frac{\Delta C_p}{4}\right)_n \sin\theta_n \left(\cos\theta_n - \cos\theta_{REF}\right).$$

NACA 4 DIGIT AIRFOIL SUBROUTINE

The NACA foils are denoted as m p tc

Maximum camber → m

Location of maximum camber → p

Maximum thickness → tc

Example: NACA 2412 → $Z_{c_{MAX}} = .02$

$X_{Z_{c_{MAX}}} = .40$

$t/c = .12$

The thickness envelope is given by:

$$Z_t = t\left[a_0\sqrt{X} - a_1 X - a_2 X^2 + a_3 X^3 - a_4 X^4\right]$$

with a leading edge radius of

$$r_{le} = 1.1019\, t^2$$

where normalization by c is understood. The values of the a's are included in the register contents list.

The camber line is:

$$Z_c = \frac{m}{p^2}\ (2pX-X^2) \qquad\qquad X \leq p$$

$$= \frac{M}{(1-p)^2}\ [(1-2p) + 2pX-X^2] \qquad\qquad X > p$$

USER INSTRUCTIONS -- PROGRAM 3.3(a)

STEP	ENTER	PRESS	DISPLAY
1. Select airfoil			
o Thickness	t/c	STO 28	
o Maximum camber	m	STO 29	
o Location of maximum camber	p	STO 30	
2. Choose the number of solution points	N	STO 00	
3. Start calculation*		A	
4. Choose angle-of-attack	α	C	$C_{P_{up}}$
		RCL 21	X_ν
5. Get lower surface pressure	-	R/S	$C_{P_{low}}$
NOTE: Repeat 4 & 5 for each α required.			
6. Go to next X	-	D	-
When calculation stops, go back to 4 and repeat 4 & 5.			

* If the printer is used, go to modified instructions.

MODIFIED INSTRUCTIONS FOR PRINTER VERSION OF 33(a)

STEP	ENTER	PRESS	DISPLAY
3. Input angle-of-attack	α	STO 20	α
4. Start calculation	-	A	Printer displays X, $C_{P_{up}}$, $C_{P_{low}}$ at each X.
Repeat 3 & 4 for each angle-of-attack.			

Note that if the printer is not used, the calculation is best performed by doing several angles-of-attack at each X, while, if the printer is used, all X's are calculated at a fixed α before going to the next α.

USER INSTRUCTIONS -- PROGRAM 3.3(b)

STEP	ENTER	PRESS	DISPLAY
1. Enter number of solution stations.	N	A	
2. If $X_{REF} \neq .25$, set X_{REF}.	X_{REF}	B	
3. Enter pressures	$C_{P_{up}}$	C	
	$C_{P_{low}}$	D	C_L
		RCL 04	C_m
Repeat 3. until all pressures are used.			

SAMPLE CASE: NACA 4412 airfoil at $\alpha = 4°$.

($t/c = .12$, $m = .04$, $p = .4$) Use $N = 16$.

From Program a)

n	X	C_{p_u}	C_{p_L}
15	.0096	- .9895	.8372
14	.0381	-1.1301	.4087
13	.0843	-1.1457	.2661
12	.1464	-1.1496	.2320
11	.2222	-1.1165	.2307
10	.3087	-1.0385	.2374
9	.4025	- .8888	.2233
8	.5000	- .7274	.2083
7	.5975	- .5919	.2076
6	.6913	- .4621	.2069
5	.7778	- .3346	.2049
4	.8536	- .2068	.2042
3	.9157	- .0754	.2071
2	.9619	.0595	.2212
1	.9904	.1628	.2025

From Program b) $C_L = .923$, $C_m = -.109$.

3.4 CAMBER LINE DESIGN USING LINEAR THEORY

This program computes the camber line required to produce a specified load distribution, ΔC_p, on an airfoil. The method is based on thin airfoil theory and uses the scheme given by C. E. Lan, "A Quasi-Vortex-Lattice Method in Thin Wing Theory," Journal of Aircraft, Vol. 11, No. 9, September 1974, pp. 518-527. It is also useful in determining the modification to a camber line required to produce the specified change in load distribution.

METHOD

The governing equation for thin wing theory is:

$$\frac{dz}{dx} = -\frac{1}{4\pi}\int_0^c \frac{\Delta C_p \, dx'}{X - X'}$$

where dz/dx includes the angle-of-attack. The original Lan theory was used to find ΔC_p (in a slightly more elaborate form), but it can also be used to obtain dz/dx from ΔC_p. To do this, the following summation is used to obtain the slope and once the slope is known, it is integrated to obtain the camber line.

Start with:

$$\frac{dz}{dx_i} = -\frac{1}{N}\sum_{K=1}^{N} \frac{\Delta C_{p_K}}{4} \frac{\sqrt{X_K (1 - X_K)}}{X_i - X_K}$$

where

$$X_K = \frac{1}{2}\left[1 - \cos\left\{\frac{(2K-1)\pi}{2N}\right\}\right] \qquad K = 1, 2, \ldots N$$

and

$$X_i = \frac{1}{2}\left[1 - \cos\left\{\frac{i\pi}{N}\right\}\right] \qquad i = 0, 1, \ldots N.$$

Here N + 1 is the number of stations on the foil at which the slopes are obtained.

Given dz/dx_i, the camber line is then computed by the trapezoidal rule (marching forward starting at the trailing edge):

$$Z_{i+1} = Z_i - \left[\frac{X_{i+1} - X_i}{2}\right]\left[\frac{dz}{dX_i} + \frac{dz}{dX_{i+1}}\right] .$$

The design angle-of-attack is then:

$$\alpha_{DES} = \tan^{-1} Z_0$$

The camber line can then be redefined in standard nomenclature; i.e., $Z(X = 0) = Z(X = 1) = 0.0$,

$$\overline{Z}_i = Z_i - (1 - X_i) \tan \alpha_{DES}.$$

Note that Lan's original equation contains a leading edge suction term. Since ΔC_p cannot be specified to be infinite, this term is not necessary. It is important to note that formally, a finite load cannot be specified at the leading edge in thin airfoil theory. This fact is reflected in an infinite slope of the camber line at the leading edge when a finite load is specified. In practice, a finite load is often specified at the leading edge even though it is not actually correct. This practice was probably accepted because the NACA 6 series camber lines employ a finite load at the leading edge.

In this program, ΔC_p is required at arbitrary locations and linear interpolation is used to obtain ΔC_p's at points other than the input locations.

USER INSTRUCTIONS -- PROGRAM 3.4

STEP	ENTER	PRESS	DISPLAY
1. Store the ΔC_p distribution (10 X, ΔC_p pairs maximum)	X_1=0.0 X_2= ⋮ X_M=1.0 ΔC_{p_1} ΔC_{p_2} ⋮ ΔC_{p_M}	STO 11 STO 12 ⋮ STO 10+M STO 21 STO 22 ⋮ STO 10+M	
2. Initialize Calculation	-	A	21.
3. Enter number of points at which camber line is desired and start calculation (N=24 maximum and 34 if the partitioning is set to 7 OP 17).	N	B $x \gtrless t$ RCL 34	$\left.\frac{dz}{dx}\right\|_{X=1.0}$ X Z
4. Go to next X station. Repeat 4. until X=0.	-	R/S $x \gtrless t$ RCL 34	dz/dx X Z
5. When X=0, compute design angle-of-attack.	-	R/S	α_{DES}
6. To change stored Z's to $\overline{Z}$'s (starting at the tail and moving forward)	-	C RCL 34 ⋮ RCL 35+N	0.0 $\overline{Z}$(X=1) ⋮ $\overline{Z}$(X=0)

If the printer is used, unlabeled output is generated as X, Z, dz/dx during the calculation. During the $\overline{Z}$ calculation, X, $\overline{Z}$ are output. Note that deleting steps 239 and 291 will eliminate the R/S commands and the program will automatically cycle all the way through the calculation.

<u>SAMPLE CASE</u>: The NACA 6 Series mean line with a = .4, C_{ℓ_i} = 1.0.

Store ΔC_p:				
X = 0.0	STO 11	ΔC_p = 1.42857	STO 21
0.4	STO 12	1.42857	STO 22
1.0	STO 13	0.0	STO 23

N = 20

	X	Z	dz/dx	$\bar{Z}$
1.	1.0000	0.0000	-.1719	0.0000
3.	.9755	.0044	-.1884	0.0030
5.	.9045	.0187	-.2102	0.0131
7.	.7939	.0425	-.2187	0.0304
9.	.6545	.0721	-.2011	0.0517
11.	.5000	.0992	-.1441	0.0698
13.	.3455	.1124	-.0153	0.0739
15.	.2061	.1071	.0911	0.0604
17.	.0955	.0912	.2041	0.0380
19.	.0245	.0717	.3709	0.0143
21.	.0000	.0635	.9806	0.0000

α_{DES} = 3.37°

For this camber line, the analytical results are available for comparison. For example, the analytical α_{DES} = 3.46°. Thus, the numerical results are within a tenth of a degree. Similarly, the camber line is within .74% of the exact result at X = .5.

3.5 WING ANALYSIS USING LIFTING LINE THEORY

This program computes the lift, drag and spanload on unswept wings using lifting line theory. The technique employed is known as Multhopp's Method and was selected because the solution matrix is well posed for iterative solution procedures. It is interesting to note that the classical "monoplane equation" (which uses Fourier analysis completely) produces a matrix which is not diagonally dominant and, hence, the monoplane equation cannot be solved by iterative methods. Primary references on Multhopp's Method are the text by R. L. Bisplinghoff, H. Ashley and R. L. Halfman, Aeroelasticity, Addison-Wesley, Reading, 1955, pp. 229-243 and "Theoretical Symmetric Span Loading at Subsonic Speeds for Wings Having Arbitrary Planform," by J. deYoung and C. W. Harper, NACA R-921, 1948. The present program uses Gauss-Seidel iteration to obtain the solution, allowing up to eight spanwise solution stations on the semi-span.

METHOD

Given the aspect ratio, AR, the taper ratio, λ, the twist, $\Delta\theta$ (as a subroutine), and the section lift curve slope, a_0, then the problem statement is completed by specifying the root angle-of-attack, α_0.

Therefore, define the span station $\eta_\nu = \cos\phi_\nu$, where $\phi = \frac{\nu\pi}{M+1}$ and a spanload variable:

$$G = \frac{1}{2AR} \cdot \frac{CC_\ell}{C_A},$$

where C_ℓ is the section lift coefficient, C is the local chord, and C_A is the average chord. M is the total number of solution stations.

The solution is obtained by solving a matrix equation written in the form:

$$(\alpha_0 + \Delta\theta_\nu) = \sum_{n=1}^{N} a_{\nu n} G_n \qquad \nu = 1, 2 \ldots N.$$

where M = 2N-1 and represents the contraction due the symmetrical loading assumption. Here:

$$a_{\nu\nu} = \frac{1+M}{4\sin\phi_\nu} + \frac{(1+\lambda)\,AR}{a_0\,[1 + (1-\lambda)\cos\phi_\nu]} \qquad \text{for } n = \nu$$

$$a_{\nu n} = -B_{\nu n} \qquad \text{for } n \neq \nu$$

where $B_{\nu n} = b_{\nu n} + b_{\nu,\ M+1-n}$ for $n \neq N$

$= b_{\nu n}$ for $n = N$

and

$$b_{\nu n} = \frac{\sin \phi_n}{[\cos \phi_n - \cos \phi_\nu]^2} \cdot \left[\frac{1 - (-1)^{n-\nu}}{2\ (M + 1)} \right].$$

The equation is then solved by iteration:

$$G_\nu^{q+1} = \frac{1}{a_{\nu\nu}} \left[(\alpha_0 + \Delta\theta_\nu) - \sum_{n=1}^{N}{}' \ a_{\nu n}\ G_n^q \right] \qquad \nu = 1. . .N$$

where the $\sum'$ denotes summation excluding $n = \nu$. In addition, Gauss-Seidel iteration implies that the latest values of G_ν are used when available (which is the natural scheme when using computer instructions). In general, 6-8 iterations are sufficient to obtain convergence.

Note that to start the solution, N must be picked. Allowable values are 4, 6 or 8. In general, a value of 4 is adequate for finding the lift and drag coefficients. The appropriate convergence tolerance must be selected. The program default is $E_{CONV} = .1 \times 10^{-5}$ and may be unnecessarily strict.

Once the G's are determined, the lift and drag are determined from:

$$C_L = \frac{\pi \cdot AR}{2N} \left[G_N + 2 \sum_{n=1}^{N-1} G_n \sin \phi_n \right]$$

and

$$C_{D_i} = \frac{\pi\ AR}{2N} \left[G_N \left(\frac{N}{2} G_N + \sum_{n=1}^{N}{}' \ B_{Nn}\ G_n \right) + 2 \sum_{\nu=1}^{N-1} G_\nu \left\{ b_{\nu\nu} \cdot G_\nu + \sum_{n=1}^{N}{}' \ B_{\nu n}\ G_n \right\} \sin \phi_\nu \right].$$

The span efficiency is then found from: $e = C_L{}^2/\pi AR\ C_{D_i}$.

Finally, note that the program is written with explicit labels for clarity, but could be rewritten using direct addressing. This would result in a significant reduction in execution time.

USER INSTRUCTIONS -- PROGRAM 3.5

STEP	ENTER	PRESS	DISPLAY
1. Define Wing*			
i) Aspect Ratio	AR	A	AR
ii) Taper Ratio	λ	B	λ
iii) Angle-of-Attack (degrees)	α_0	C	$a_0=2\pi$ (default)
iv) Section lift curve slope if not 2π (in $radian^{-1}$)	a_0	E'	a_0
2. Choose convergence criterion if not default ($.1 \times 10^{-5}$)	E_{CONV}	D'	E_{CONV}
3. Choose number of points on semi-span (4, 6 or 8)	N	D RCL 19	C_L C_{D_i}
4. Get spanload (if desired)	-	C' $X \gtrless t$	η_ν CC_ℓ/C_A
NOTE: Repeat 4. N times.			

* Twist subroutine must be supplied if wing has twist: Label E, η_ν is in display and $\Delta\theta_\nu$ in radians is returned.

If the printer is used, the G's at each iteration are printed as they are computed, followed by E after each iteration is completed. Once the solution has converged, C_{D_i} and C_L are printed, all as unlabeled output.

SAMPLE CASE: $AR = 10$

$\lambda = .6$

$\alpha_0 = 10°$

N = 4

	η_ν	CC_ℓ/C_A
4	0.000	1.1314
3	0.383	1.0081
2	0.707	0.8278
1	0.924	0.5656

C_{D_i} = .02664

C_L = .90275

N = 6

	η_ν	CC_ℓ/C_A
6	0.000	1.1263
5	0.259	1.0576
4	0.500	0.9506
3	0.707	0.8274
2	0.866	0.6740
1	0.966	0.4296

C_{D_i} = .02664

C_L = .90091

N = 8

	η_ν	CC_ℓ/C_A
8	0.000	1.1239
7	0.195	1.0791
6	0.383	1.0059
5	0.556	0.9214
4	0.707	0.8273
3	0.831	0.7172
2	0.924	0.5700
1	.981	0.3431

C_{D_i} = .02661

C_L = .90012

N	Computing Time (TI59)
4	33 min. (8 iterations)
6	100 min. (11 iterations)
8	4 hr. (13 iterations)

NOTE: A reduced tolerance criterion (E_{CONV} increased) would reduce execution time without an appreciable change in the answer. Some experimentation on the part of the user is required.

3.6 INDUCED DRAG ANALYSIS

For a given spanload, this program determines the lift induced drag. The method is valid for a single planar lifting surface or two surfaces if they are in the same plane and the spanload is the sum of the spanloads. The analysis also gives the lift coefficient. It is assumed that the spanload is symmetric.

The method is based on a Fourier Series representation of the spanload, and thus, is based on the analysis due to Glauert. The book by Houghton, E. L. and Brock, A. E., Aerodynamics For Engineering Students, Edward Arnold, London, 1960, pp. 335-369, provides a good presentation of the Fourier Methods for lifting line theory.

It is important to note that the method actually computes the span "e" and not the drag coefficient. In order to compute C_D, the Aspect Ratio must be specified. However, it is important to remember that the actual drag is inversely proportional to the span squared and not the Aspect Ratio. In addition, for arbitrary wings, "e" is not a constant, but may vary with lift coefficient. This fact has not been brought out clearly in several recent texts.

METHOD

Assuming that the spanload is described by a Fourier Series:

$$\frac{CC_\ell}{C_A} = \sum_{n=1}^{N} a_n \sin\left[(2n-1)\theta\right],$$

then the a_n's are given by

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} \frac{CC_\ell}{C_A}(\theta)\sin\left[(2n-1)\theta\right]\, d\theta$$

The spanwise location $\eta = y/(b/2)$ is related to θ by $\eta = \cos\theta$.

The rest of the results are then computed from:

$$C_L = \frac{\pi}{4}\cdot a_1$$

$$e = \frac{1}{\sum_{n=1}^{N}(2n-1)\left(\frac{a_n}{a_1}\right)^2} \quad \text{and} \quad C_{D_i} = \frac{C_L^{\,2}}{\pi ARe}.$$

The numerical operation then consists mainly in integrating for the a_n's. Although many methods have been applied to this problem, a simple trapezoidal rule and 4 or 5 terms in the series are usually more than sufficient to provide a good estimate of e.

The first step in the procedure is to provide a means of interpolating the prescribed spanload for arbitrary values of θ. In this particular calculator program, a linear interpolation is used and this representation (with a maximum of 15 values of spanload) restricts the Fourier Series analysis to 5 or 6 terms. For more terms, a better interpolation should be used.

To carry out the integration, the trapezoidal rule is used:

$$I = \int_a^b f(X)\,dX = \Delta X \left\{ f(a + \Delta X) + \ldots + f(b - \Delta X) + \frac{f(b)}{2} \right\}$$

where a zero loading on the tip has been assumed.

A Δθ is picked which depends on the term in the series:

$$\Delta\theta = \frac{\pi/2}{Q}$$

where

$$Q = A(2N - 1) - (N - n)$$

and A - number of trapezoidal panels in 1/2 cycle of N.

N - number of terms in series.

Typical values of A are 3 or 4, while N = 4 is usually satisfactory.

Finally, note that $C_A = S_{REF}/b$ and is the average chord. This is the normalization that is employed because it leads to the simple result:

$$C_L = \int_0^1 \frac{cc_\ell}{C_A}(\eta)\,d\eta\,,$$

which is valid for the symmetrical spanload case.

SAMPLE CASES

1. Elliptic Spanload:

$$\frac{cc_\ell}{c_A} = \sqrt{1 - \eta^2}$$

A = 4

N = 4

	η	$\frac{cc_\ell}{c_A}$
1.	0.00	1.0
2.	0.20	0.9798
3.	0.40	0.9165
4.	0.60	0.8000
5.	0.80	0.6000
6.	0.90	0.4359
7.	0.96	0.2800
8.	1.00	0.0000

Result:

$a_4 = -.0115$

$a_3 = -.0096$

$a_2 = -.0108$

$a_1 = .9907$

$e = .998$

$C_L = .778$

Where the analytic result is $a_1 = 1$, all others zero, $e = 1.0$.

2. Linear Spanload: $\frac{cc_\ell}{c_A} = 1 - \eta$

A = 4

N = 4

	η	$\frac{cc_\ell}{c_A}$
1.	0.0	1.0
2.	1.0	0.0

Results:

	Numerical	Analytical
a_4	-.0307	-.0303
a_3	.0428	.0424
a_2	-.2126	-.2122
a_1	.6370	.6366

$e = .728$

$C_L = .500$