MASON

AERODYNAMIC CALCULATION METHODS for PROGRAMMABLE CALCULATORS & PERSONAL COMPUTERS

— with programs for the TI-59 —

PAK #4
BOUNDARY LAYER ANALYSIS METHODS

by W.H. MASON

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AEROCAL

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Preface

AEROCAL programs are intended to serve both students and practicing aerodynamicists. For students, they can serve an important role in supplementing theoretical analysis with the actual numerical results so important in developing engineering skills. In aerodynamics, it has been difficult for students to solve meaningful illustrative problems, and this difficulty can now be eliminated by using the new personal computing machines -- either programmable calculators or microcomputers. I have found that the results of numerical calculations inevitably provide a few surprises, which force the analyst to reexamine the theory, leading to a much deeper understanding. AEROCAL programs can thus be used to prevent the calculation from becoming an end in itself. Instead, efforts can be concentrated on the actual aerodynamic problems, with required calculations assuming their proper supporting role. Thus, the availability of the personal computing machines allows the student to gain an appreciation of the role of computational aerodynamic simulations, while developing an engineering attitude.

The second purpose of the work is to provide the practicing aerodynamicist with a readily accessible collection of algorithms designed for use on this class of machine. The availability of such a set of routines will eliminate the most tedious aspects of the software development process so that the code development time can be used to implement the user's unique requirements rather than wasting time creating the basic building blocks.

The material selected for inclusion is, of course, not intended to replace the large computational aerodynamics programs. Instead, it allows students to become familiar with an important part of the set of standard aerodynamic methods representative of those required in aerodynamics. To the experienced user, these methods should be extremely useful, providing results which are more than adequate for a variety of jobs.

The material is organized in workbook fashion, with each program being essentially independent of the others. An example of the style that we intend to follow is found in the IBM SSP or other software package user's manuals. The addition of some examples for each program allows the user to check that the program is properly executing on his own machine.

The choice of the TI59 format for the programs is one of convenience only. Program instructions are similar for other calculators and an Appendix is included to describe the listing nomenclature. Using this information, conversion to other instruction sets should be relatively simple. Microcomputers will typically have more advanced instruction sets, such as BASIC. The information provided in the method description is easily used to write a set of BASIC instructions.

The author acknowledges the contributions of the many aerodynamicists and research scientists who have developed the basic material, which forms the basis for these software paks and with whom he has held discussions on the relative merits of particular methods for performing various aerodynamics calculations.

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About the Author

W. H. Mason has spent more than ten years developing and applying computational aerodynamics methodology to transonic and supersonic aircraft design. This work required the use of the full range of computer codes presently used in the industry, so that the author has an unusually broad base of experience to draw upon. He obtained the B.S., M.S. and Ph.D. degrees in Aerospace Engineering at Virginia Polytechnic Institute and is presently employed as a Senior Engineer in the Aerodynamics Section of Grumman Aerospace Corporation. Dr. Mason is a registered Professional Engineer in New York State.

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4.0 INTRODUCTION

Estimates for skin friction, flow separation and transition location can be made just as easily on a programmable calculator as on a large computer. For these classes of flows, the methods used in industry for aerodynamics work are relatively simple. You cannot do much better, even with a larger computing machine. More elaborate methods are not yet in general use and require much more validation before acceptance. This surprising state of affairs is due to the need for a better understanding of the physics of turbulent flow details and not larger computers, in order to advance the state-of-the-art in practical turbulent flow prediction methodology. The methods presented in this pak are, however, mainly limited to 2-D incompressible flows. Extension to include axisymmetric cases and compressibility effects are straightforward and can be carried out using microcomputers slightly larger than the TI59.

The choice of methods is based on typical problems in aircraft aerodynamics and methods which have been found to supply reliable answers to these problems. Simple flat plate skin friction estimates provide the basis for most friction estimates. The detailed boundary layer distribution calculations are performed to determine if separation occurs, to provide information for transition grit location when simulating full scale viscous effects at reduced Reynolds numbers, or to determine the displacement thickness for use in correcting the inviscid flow calculation.

In normal aircraft applications, the boundary layer is mostly turbulent and the simple method of Thwaites is more than sufficient. For example, one of the widely used transonic airfoil programs (KORN) does not even compute for the laminar flow portion of the boundary layer. Although two schemes for predicting transition are included, it is fair to say that no methods are generally trusted for accurate estimates of transition location, especially for mostly favorable pressure gradient cases.

Two methods for computing turbulent boundary layers are included. Although they are both integral methods (and thus not considered the most advanced techniques), they were developed for aerodynamic pressure gradients and, in the case of Green's Method, known to be very accurate. Once again, note that both major 2-D transonic codes use integral methods.

The effects of transition position and compressibility are treated in separate methods, and finally, the Squire-Young formula is included. This is the standard formula for computing the actual airfoil profile drag, which includes both friction and form (or pressure) drag.

The methods are presented in a standard format with the following information:

- Title
- Description of what the method does
- References
- Detailed outline of the method and listing of the equations required
- User instructions
- Sample case
- Program description
- Program listing.

The programs are written in the most direct sequence of instructions possible in order to make the study of the programs as simple as possible. This allows the user to incorporate modifications to the programs or convert them to other systems without difficulty. The use of the TI59 instruction set is purely a convenient selection. These routines will work on a number of other calculators, as well as the emerging class of microcomputers. For those readers not familiar with the details of the TI59 instruction set, a description is included in Appendix A. This will allow the non-TI59 user to convert the codes to his own instruction set with ease.

The routines often make use of the printer. The author has found the printer to be much more valuable in program development than in program execution. Nevertheless, several programs do provide printed results. A description of the printed output is included below the user instructions for each program.

4.1 INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS -- THWAITES' METHOD

The laminar boundary layer development can be calculated approximately using a simple integral to obtain the momentum thickness and then finding the skin friction and displacement thickness from correlation formulas that depend on the momentum thickness and pressure gradient. This method is generally known as Thwaites' Method and is especially well suited to computing the laminar portion of the boundary layer in cases where transition occurs and the main part of the boundary layer is turbulent. This method then serves to provide the initial conditions for the turbulent boundary layer calculation. A discussion of the errors arising from the use of this method can be found in the book by N. Curle, The Laminar Boundary Layer Equations, Clarendon Press, Oxford, 1962. He cites errors of up to 5% in momentum thickness and 10% in displacement thickness.

This program uses Thwaites' Method to compute the laminar boundary layer. For more information, two more references are especially helpful:

Cebeci, T. and Bradshaw, P. Momentum Transfer in Boundary Layers. Hemisphere Publishing Corp., Washington, 1977, pg. 108-112; 240.

Rott, N. and Crabtree, L. F., "Simplified Laminar Boundary Layer Calculations for Bodies of Revolution and For Yawed Wings," J. Aero. Sci., 19, 553-65 (1952).

METHOD

Given the Reynolds number, $Re_L = \frac{\rho U_\infty L}{\mu}$ and the surface velocity, $U_e(X)/U_\infty$, the momentum thickness, (θ/L) is found from:

$$\left(\frac{\theta}{L}\right)^2 = \frac{.45}{Re_L\left(\frac{U_e}{U_\infty}\right)^6}\int_{(X_0/L)}^{(X/L)}\left(\frac{U_e}{U_\infty}\right)^5 d\left(\frac{X}{L}\right) + \left(\frac{\theta_0}{L}\right)^2\left(\frac{U_{e_0}/U_\infty}{U_e/U_\infty}\right)^6$$

where (X/L) is the non-dimensional distance from the origin, measured along the surface. θ_0 and U_{e_0} are the values at $X = X_0$. In most aerodynamic calculations $X_0 = 0$ and $\theta_0 = 0$ (flat plate) or $U_{e_0} = 0$ (stagnation point) and the last term is zero.

Note that at a stagnation point, the first term can be evaluated by performing the integration first and then cancelling terms; i.e.:

$$\frac{U_e}{U_\infty}(X) = \frac{d(U_e/U_\infty)}{d\,(X/L)} \cdot X \longrightarrow \left(\frac{\theta_s}{L}\right)^2 = \frac{.075}{Re_L \cdot \frac{d(U_e/U_\infty)}{d(X/L)}} \quad .$$

Once the momentum thickness is known, the rest of the boundary layer information can be found using the parameter:

$$\lambda = \left(\frac{\theta}{L}\right)^2 \mathrm{Re}_L \frac{d(U_e/U_\infty)}{d(X/L)}$$

Note that λ depends on the velocity gradient. Although θ can be computed without this information, it is needed to find the skin friction and displacement thickness.

λ is also used to determine the separation point and when λ is less than λ_{SEP}, separation has occurred. Thus, an important value is:

$$\lambda_{SEP} = -.09$$

Once λ is known, the ℓ and H functions are computed using curve fits given by Cebeci and Bradshaw:

For $0 \leq \lambda \leq 0.1$: $\ell = .22 + 1.57\,\lambda - 1.8\,\lambda^2$

$H = 2.61 - 3.75\,\lambda + 5.24\,\lambda^2$

and for $-.1 \leq \lambda \leq 0$: $\ell = .22 + 1.402\,\lambda + .018\,\lambda/(.107 + \lambda)$

$H = .0731/(.14 + \lambda) + 2.088$

H is the shape factor and the displacement thickness is found from:

$$\frac{\delta^*}{L} = H \cdot \left(\frac{\theta}{L}\right) .$$

The local skin friction is given by:

$$C_{fe} = \frac{2\ell}{\mathrm{Re}_\theta}$$

where $C_{fe} = \frac{2\tau_W}{\rho U_e^2}$ and $\mathrm{Re}_\theta = \frac{\rho U_e \theta}{\mu}$. The skin friction based on freestream is $C_{f_\infty} = C_{fe}\left(\frac{U_e}{U_\infty}\right)^2$. For use in transition estimates, the program computes another Reynolds number, $\mathrm{Re}_X = \rho U_e X/\mu$.

In order to carry out the calculation, a routine must be supplied which provides $U_e(X)/U_\infty$ and $d(U_e(X)/U)/d(X/L)$.

Some useful relations between C_p and U_e/U_∞ are:

$$\frac{U_e}{U_\infty} = \sqrt{1 - C_p} \;; \qquad \frac{d(U_e/U_\infty)}{d(X/L)} = -\frac{1}{2}\,\frac{1}{(U_e/U_\infty)}\cdot\frac{dC_p}{d(X/L)} \;.$$

In the program listing, the model inviscid flow routine is for a circular cylinder. In this case, it is useful to use the cylinder radius as the reference length and it can be shown that in the boundary layer coordinate system:

$$\frac{U_e}{U_\infty} = 2\sin\left(\frac{X}{r_0}\right) \;; \qquad \frac{dU_e/U_\infty}{d(X/r_0)} = 2\cos\left(\frac{X}{r_0}\right) \;.$$

For a step by step method of specifying the external flow, see the approach used in Program 4.3.

Finally, note that for the TI-59 listing, ML-09, Simpson's Rule for integration is used in computing the integral. If another machine is used, the trapezoidal rule can be used with both accuracy and simplicity.

$$\Delta I = \int_{X_{i-1}}^{X_i} f(\bar{X})\,d\bar{X} = \left(X_i - X_{i-1}\right)\left[\frac{f(X_i) + f(X_{i-1})}{2}\right]$$

where for relatively small step sizes, no intermediate steps are required in the integral evaluation.

USER INSTRUCTIONS -- PROGRAM 4.1

STEP	ENTER	PRESS	DISPLAY
1. Select step size for integration.	$\left(\frac{\Delta X}{L}\right)_{NOMINAL}$	A	0
2. Input Reynolds number (Note: Include exponent.)	Re_L	B	Re_L
If $X_0 = 0$, skip 3. and 4. (i.e., $\theta_0 = 0$ or $U_{e_0} = 0$).			
3. Input $\left(\frac{X_0}{L}\right)$	$\left(\frac{X_0}{L}\right)$	C'	$d(U_e/U_\infty)/d(X/L)\|_{X_0}$
4. Input $\left(\frac{\theta_0}{L}\right)$	$\left(\frac{\theta_0}{L}\right)$	E'	$\left(\frac{\theta_0}{L}\right)^2\left(\frac{U_{e0}}{U_\infty}\right)^6$
5. Input station at which solution is desired. [Note: (X/L) must be at least 2(ΔX/L) greater than the previous X/L.]	$\left(\frac{X}{L}\right)$	C	Re_θ*
		RCL 17	θ/L
		RCL 12	H
		RCL 7	δ*/L
		RCL 6	C_{f_e}
		RCL 20	C_{f_∞}
Repeat 5. at each value of X/L.		RCL 22	Re_X
		RCL 14	X/L
		RCL 10	λ
6. To reset (ΔX/L) at any step in the calculation, input the new ΔX/L before Step 5. is repeated.	$\left(\frac{\Delta X}{L}\right)_{NOMINAL}$	B'	$\left(\frac{\Delta X}{L}\right)$

* Display flashes when separation is predicted and none of the results are computed at this X.

SAMPLE CASE: Circular cylinder ($L = r_0$)

$\frac{\Delta X}{L} = .04$, $Re_L = 1 \times 10^6$

θ°	X/r_0*	λ	θ/L	δ*/L	C_{f_∞}
10	.1745	.0753	.000195	.000461	.001165
20	.3491	.0738	.000198	.000468	.002251
40	.6981	.0695	.000213	.000506	.003868
60	1.0472	.0589	.000243	.000584	.004371
80	1.3963	.0311	.000299	.000748	.003515
90	1.5708	.0000	.000346	.000904	.002540
95	1.6581	-.0249	.000378	.001029	.001895
100	1.7453	-.0602	.000417	.001252	.001062
103	1.7977	-.0888	.000444	.001562	.000066
106	1.8500	FLASHING DISPLAY → SEPARATION			

Change Step Size

$\frac{\Delta X}{L} = .02$

$* \frac{X}{r_0} = \frac{\pi}{180} \cdot \theta(°)$

Note that this method predicted separation at past θ = 103°, but before θ = 106°, which agrees with results found by other methods.

DETAILS OF PROGRAM 4.1

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-11	Routine A, save ΔX, initialize $X_{i-1}=0=\theta_0 \cdot U_{e_0}$	0	-
12-18	Routine B, save Re_L	1-5	Reserved for ML-09
19-23	Routine B', new ΔX	6	C_{f_e}
24-29	Routine C', store Xi/L new and find U_e/U_∞	7	δ^*/L
30-45	Routine E', input θ_0 and compute R_{13}	8	$\Delta X/L_{NOMINAL}$
46-55	Routine C, store new X_i, X_{i-1}	9	Re_L
56-76	Find exact value of ΔX and number of integration steps	10	λ
77-82	Compute the integral from X_{i-1} to X_i using Simpson's rule routine in Master Library	11	ℓ
83-84	Add new increment to previously computed part of integral	12	H
85-118	Compute and store θ/L	13	$(\theta_0/L)^2 \cdot (U_{e_0}/U_\infty)^6$
119-131	Compute and store λ	14	X_i
132-138	Test for boundary layer separation	15	(U_e/U_∞)
139-140	Call routine to compute ℓ and H	16	$d(U_e/U_\infty)/d(X/L)$
141-149	Compute δ^*/L	17	θ/L
150-166	Compute C_{f_e}	18	I (the integral)
167-170	Update X_{i-1} to latest X	19	X_{i-1}
171-180	Compute C_{f_∞}	20	C_{f_∞}
		21	X used in E; Re_θ
		22	Re_X

DETAILS OF PROGRAM 4.1

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-11	Routine A, save ΔX, initialize $X_{i-1}=0=\theta_0 \cdot U_{e_0}$	0	-
12-18	Routine B, save Re_L	1-5	Reserved for ML-09
19-23	Routine B', new ΔX	6	C_{f_e}
24-29	Routine C', store Xi/L new and find U_e/U_∞	7	δ^*/L
30-45	Routine E', input θ_0 and compute R_{13}	8	$\Delta X/L_{NOMINAL}$
46-55	Routine C, store new X_i, X_{i-1}	9	Re_L
56-76	Find exact value of ΔX and number of integration steps	10	λ
77-82	Compute the integral from X_{i-1} to X_i using Simpson's rule routine in Master Library	11	ℓ
83-84	Add new increment to previously computed part of integral	12	H
85-118	Compute and store θ/L	13	$(\theta_0/L)^2 \cdot (U_{e_0}/U_\infty)^6$
119-131	Compute and store λ	14	X_i
132-138	Test for boundary layer separation	15	(U_e/U_∞)
139-140	Call routine to compute ℓ and H	16	$d(U_e/U_\infty)/d(X/L)$
141-149	Compute δ^*/L	17	θ/L
150-166	Compute C_{f_e}	18	I (the integral)
167-170	Update X_{i-1} to latest X	19	X_{i-1}
171-180	Compute C_{f_∞}	20	C_{f_∞}
		21	X used in E; Re_θ
		22	Re_X

DETAILS OF PROGRAM 4.1 (Continued)

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
181-192	Compute $Re_X = U_e X_i/\nu$		
193-206	Compute $Re_\theta = U_e \theta/\nu$		INVISCID FLOW ROUTINE
207-214	Routine π, start ℓ, H calculation, test if $\lambda \leq 0$	1.	Label E.
215-237	For $\lambda > 0$, compute ℓ	2.	X is in display when called and must be stored in R_{21}.
238-263	For $\lambda > 0$, compute H	3.	Compute U_e/U_∞ and store in R_{16}
264-299	For $\lambda < 0$, compute ℓ		
300-324	For $\lambda < 0$, compute H	4.	Compute $\frac{dU_e/U_\infty}{dX/L}$
325-333	Routine A', integrand for integration subroutine		store in R_{15} and leave in display on return.
334-338	Routine EE, causes display to flash, indicating separation	NOTE:	Leave calculator in DEG mode on return.
339-363	Routine E, inviscid flow routine		

4.2 TRANSITION LOCATION: TWO APPROXIMATE METHODS

There are no really reliable simple methods to predict boundary layer transition location. However, two methods have been cited by T. Cebeci and A. M. O. Smith in Analysis of Turbulent Boundary Layers, Academic Press, New York, 1974, pp. 332-335 and are appropriate for "small calculations." The present program (designed to work with Program 4.1) performs the transition prediction calculations of Michel and Granville. In these methods, $Re_\theta \equiv U_e\theta/\nu$ and $Re_X \equiv U_e X/\nu$.

a) Michel's Method

At each station, compute $Re_{\theta tr}$ for the known Re_X. If the local Re_θ is greater than $Re_{\theta tr}$, then transition has occurred. The Michel formula given by Cebeci and Smith is:

$$Re_{\theta tr} = 1.174 \left[1 + \frac{22,400}{Re_X}\right] Re_X^{.46}$$

This relation is valid for $.1 \times 10^6 \leq Re_X \leq 40 \times 10^6$.

b) Granville's Method

Initially at each X station, compute $Re_{\theta i}$ based on H. $Re_{\theta i}$ is the value of boundary layer instability and the formula given by Cebeci and Smith is:

$$Re_{\theta i} = e^{\sum_{n=0}^{7} \overline{C}_n H^n}$$

which is a curve fit for $2.45 < H < 3.4$ and the values of the constants are given in the program description, registers 23-30. If the local Re_θ exceeds $Re_{\theta i}$, then the local Re_θ should be saved as $Re_{\theta i}$, along with Re_X as Re_{Xi} and U_e/U_∞ as $(U_e/U_\infty)_i$.

Once the point of instability has been determined and the values at that point saved, the calculation at each station changes. Now, at each X station, the value of $\overline{\lambda}_\theta$ is computed from:

$$\overline{\lambda}_\theta = \frac{4}{45} - \frac{1}{5}\left[\frac{Re_\theta^{\,2} - \left(\frac{U_e}{U_{ei}}\right)^2 Re_{\theta i}}{Re_X - \left(U_e/U_{ei}\right) Re_{Xi}}\right] .$$

Then the transition increment is computed based on $\bar{\lambda}_\theta$:

$$\left(Re_{\theta tr} - Re_{\theta i}\right) = \sum_{n=0}^{4} C_n \bar{\lambda}_\theta^{\,n}$$

for $-.04 \leq \bar{\lambda}_\theta \leq .024$. The values of the constants are given in the program description, registers 23-30. With $(Re_{\theta tr} - Re_{\theta i})$ known, a test is made:

$$\left(Re_\theta - Re_{\theta i}\right) > \left(Re_{\theta tr} - Re_{\theta i}\right) \quad .$$

If this inequality is true, then transition has taken place according to Granville's Method.

USER INSTRUCTIONS -- PROGRAM 4.2

STEP	ENTER	PRESS	DISPLAY
USED WITH PROGRAM 4.1			
a) Michels Method: After each X step in Program 4.1, test for transition.	-	D	0 - no transition 1 - transition occurred
b) Granville's Method*: After each X step in Program 4.1, test for transition.	-	D'	0 - no transition 1 - transition occurred
USED WITHOUT PROGRAM 4.1			
a) Michel's Methods			
i) Store Re_X	Re_X	STO 22	
ii) Store Re_θ	Re_θ	STO 21	
iii) Test	-	D	0 - no transition 1 - transition occurred
b) Granville's Method*			
i) Store values	H	STO 12	
	U_e/U_∞	STO 15	
	Re_θ	STO 12	
	Re_X	STO 22	
ii) Test	-	D'	0 - no transition 1 - transition occurred
Repeat i) and ii) until transition (can never occur on first test).			

* For repeat cases: INV St. flag 1 to lower flag 1. This is required because flag 1 is left raised at finish of execution.

SAMPLE CASE: Flat Plate ($U_e/U_\infty = 1$, $dU_e/dX = 0.0$)

$Re_L = 10 \times 10^6$

TRIAL NO. 1	X/L	MICHEL	GRANVILLE
$\left(\frac{\Delta X}{L} = .005\right)$	.02	0	0
	.06	↓	↓
	.10	↓	↓
	.14	↓	↓
	.16	0	↓
	.18	1	↓
	.20	↓	↓
	.24	↓	↓
	.28	↓	↓
	.32	↓	↓
	.34	↓	0
	.36	1	1

COMMENTS:

1. The differences between the methods indicates the uncertainty in the transition prediction methods.
2. This also indicates how favorable and zero pressure gradients represent the most difficult cases. Adverse pressure gradients lead to rapid transition and the methods can generally predict transition closer to its actual location.
3. This example demonstrates another aspect of Granville's Method. Re_{θ_i} may be small, so that the transition checks should begin near the beginning of the flow. The next example demonstrates this.

TRIAL NO. 2	X/L	GRANVILLE
$\Delta X/L = .001$	.004	0
	.006	↓
	.008*	↓
$\Delta X/L = .01$	.220	↓
	.240	0
	.260	1

* At this point Re_{θ_i} was found. To determine when Re_{θ_i} is determined, set R_{39} to zero before starting the calculation. Then check R_{39} after each test. If $R_{39} \neq 0$, then Re_{θ_i} has been found and stored. Special care in the calculation led to a 10% L shift in the transition estimate for this case.

DETAILS OF PROGRAM 4.2

STEPS	PROGRAM DESCRIPTION
0-363	Laminar B.L. Program (4.1)
364-393	Routine D, Michel's Method, compute $Re_{\theta_{tr}}$ and place in t register
394-397	Test if $Re_\theta > Re_{\theta_{tr}}$, if so, go to $\bar{X}$
398-399	No transition, set display to 0
400-403	$\bar{X}$, transition; set display to 1
404-439	Routine SIN, computes $\bar{\lambda}_\theta$ for Granville's Method
440-485	Routine COS, computes Re_{θ_i} for Granville's Method
486-562	Routine TAN, computes $Re_{\theta_{tr}} - Re_{\theta_i}$ for Granville's Method
563-564	Routine D', Granville's Method
565-567	If Re_{θ_i} has already been computed, flag 1 is raised and the Re_{θ_i} calculation is bypassed
568-576	Compute Re_{θ_i} and test to see if $Re_\theta > Re_{\theta_i}$, if so, transfer to INT. If not, set display to zero and return.
577-602	When $Re_\theta > Re_{\theta_i}$, store Re_{θ_i}, $Re^2_{\theta_i}/(U_e/U_\infty)$ and $Re_{x_i}/(U_{e_i}/U_\infty)$. Raise flag 1.
603-617	For $Re_\theta > Re_{\theta_i}$, compute $\bar{\lambda}_\theta$, $(Re_{\theta_{tr}} - Re_{\theta_i})$ and test if $(Re_\theta - Re_{\theta_i}) > (Re_{\theta_{tr}} - Re_{\theta_i})$
618-620	If test is not met, no transition; set display to 0 and return. If test is met, go to $\bar{X}$ to set display to 1 and return (transition has occurred)

R	REGISTER CONTENTS
23	$\bar{C}_0$ = 493.906
24	$\bar{C}_1$ = - 407.017
25	$\bar{C}_2$ = 53.9041
26	$\bar{C}_3$ = 24.1642
27	$\bar{C}_4$ = - 0.104478
28	$\bar{C}_5$ = - 2.814540
29	$\bar{C}_6$ = 0.355919
30	$\bar{C}_7$ = 0.012027
31	C_0 = 820.571
32	C_1 = 28273.8
33	C_2 = 707219.
34	C_3 = 5167690.
35	C_4 = -22302300.
36	$\bar{\lambda}_\theta$
37	$Re^2_{\theta_i}/(U_e/U_\infty)$
38	$Re_{x_i}/(U_{e_i}/U_\infty)$
39	Re_{θ_i}
	(Registers 1 and 2 are used for counters.)

4.3 INCOMPRESSIBLE TURBULENT BOUNDARY LAYER CALCULATION METHODS

There are many methods for calculating turbulent boundary layers. An exact method is not available because turbulent flows are extremely difficult to model numerically. The two methods presented here are typical of the so-called integral methods, wherein some analytical procedures have been carried out before the numerical problem is posed. These methods are reasonably accurate and, in fact, the second method, Green's "Lag Entrainment," is as reliable as the most sophisticated methods for airfoil type pressure distributions. Both methods presented here are very similar, with the first being faster and occupying less space at the expense of some accuracy. They are described in detail in "Calculation of Turbulent Flows," by W. C. Reynolds and T. Cebeci, which appears in Turbulence, Ed. by P. Bradshaw, Springer-Verlag. Topics in Applied Physics Series, Vol. 12, 2nd and Corrected Edition, Berlin, 1978, pp. 208-212.

In order to start the calculation, the initial values of the momentum thickness, θ/L, and shape factor, $H = \delta^*/\theta$, (δ^* is the displacement thickness) are required at the beginning of the turbulent flow region, X_0/L. These initial values are usually obtained by assuming that the momentum thickness is given by the laminar value and the shape factor decreases by about 1.2 from its laminar value:

$$\left.\frac{\theta}{L}\right|_{TURB} = \left.\frac{\theta}{L}\right|_{LAM} \qquad \text{at } X = X_0 \; .$$

$$H_{TURB} = H_{LAM} - 1.2 \qquad \text{at } X = X_0 \; .$$

An increment could be added to the momentum thickness to account for the presence of transition grit on airfoils tested in wind tunnels. However, this increment is usually neglected in airfoil calculations. This is due both to the uncertainty in estimating the size of the increment and experience, which indicates that neglecting this increment does not lead to large errors in the results calculated downstream.

Green's Method requires an additional starting value because three ordinary differential equations are required to obtain results. It is assumed that the rate of change of the entrainment parameter F is equal to its equilibrium value (an equilibrium flow is one in which the shape of the velocity and shear stress profiles do not vary with X). The equilibrium value is given as an algebraic function of θ and H and, hence, with this assumption, no additional fundamental information needs to be supplied.

ARBITRARY EDGE CONDITIONS

For boundary layer calculations, it is usually sufficient to specify a piecewise linear edge velocity distribution, such that on each segment:

$$\frac{U_e}{U_\infty} = \frac{U_e}{U_\infty}\bigg|_{E_{i-1}} + \frac{dU_e/U_\infty}{dX/L}\left(\frac{X}{L} - \frac{X}{L}\bigg|_{E_{i-1}}\right)$$

and

$$\frac{dU_e/U_\infty}{dX/L} = \text{Constant.}$$

Note that $|_{E_{i-1}}$ refers to the external flow steps and not the boundary layer. In most cases, only 2 or 3 line segments are sufficient to describe the inviscid flow.

The velocity can be obtained from the pressure coefficient, C_p, by:

$$\frac{U_e}{U_\infty} = \sqrt{1 - C_p} \qquad \frac{d(U_e/U_\infty)}{d(x/L)} = -\frac{1}{2}\frac{1}{(U_e/U_\infty)} \cdot \frac{dC_p}{d(x/L)}$$

Thus, some relatively simple calculations must be made before the boundary layer calculation is started. The user may change this approach using the information contained in the program description concerning the integration of the inviscid flow module into the main program.

a) Head's Method

Two ordinary differential equations must be solved. The first is the momentum integral equation:

$$\frac{d(\theta/L)}{d(X/L)} = \frac{C_f}{2} - (H + 2)\left(\frac{\theta}{L}\right)\left\{\frac{1}{(U_e/U_\infty)} \cdot \frac{d(U_e/U_\infty)}{d(X/L)}\right\}$$

and the second is the entrainment equation (written in terms of H):

$$\frac{dH}{d(X/L)} = \frac{1}{(\theta/L)} \cdot \frac{1}{dH_1/dH} \cdot \left\{ F - H_1\left(\frac{\theta}{L} \cdot \frac{1}{(U_e/U_\infty)} \cdot \frac{d(U_e/U_\infty)}{d(X/L)} + \frac{d(\theta/L)}{d(X/L)}\right)\right\}.$$

The algebraic relations required to completely specify the problem are:

i) The skin friction law (Ludwieg-Tillmann):

$$C_f = \frac{.246 \cdot 10^{-.678H}}{Re_\theta^{\ .268}}$$

where $Re_\theta = U_e\theta/\nu$ and C_f is based on the local edge velocity.

ii) The mass flow shape factor $H_1 = \frac{\delta - \delta^*}{\theta}$ in terms of H:

for $H \leq 1.6$

$$H_1 = .8234\ (H - 1.1)^{-1.287} + 3.3$$

$$\frac{dH_1}{dH} = -1.0597\ (H - 1.1)^{-2.287}$$

and for $H \geq 1.6$

$$H_1 = 1.5501\ (H - .6778)^{-3.064} + 3.3$$

$$\frac{dH_1}{dH} = -4.7495\ (H - .6778)^{-4.064}$$

iii) The entrainment parameter:

$$F = .0306\ (H_1 - 3.0)^{-.6169}\ .$$

In using integral methods, flow separation is determined by the value of H. Generally, separation has occurred if H is between 1.8 and 2.4. Near separation H increases rapidly and the lack of a precise value of H_{SEP} does not affect the separation point location prediction.

The details of the numerical solution are contained in the "Integration of the Ordinary Differential Equations" section which follows the description of Green's Method. The details of the edge velocity specification then follow that section and complete the information required to perform the calculation.

USER INSTRUCTIONS -- PROGRAM 4.3(a)

STEP	ENTER	PRESS	DISPLAY
1. Initial Conditions			
i) X_0/L	X_0/L	STO 03	
ii) Re_L	Re_L	STO 01	
iii) θ/L at X_0	θ/L	STO 04	
iv) H at X_0	H	STO 05	
2. Select step size, h	h	STO 00	
3. Set External Flow+			
i) $X/L\|_{E_{i-1}}$	$X/L\|_{E_{i-1}}$	STO 37	
ii) $U_e/U_\infty\|_{E_{i-1}}$	$U_e/U_\infty\|_{i-1}$	STO 35	
iii) $d(U_e/U_\infty)/d(X/L$	$d(U_e/U_\infty)/d(X/L)$	STO 36	
4. Select station at which solution is desired X/L*	$X/L\|_{NOM}$	A	X/L
		RCL 27	U_e/U_∞
		RCL 04	θ/L
		RCL 05	H
Repeat Steps 3. and 4. (to get C_f) for each station at which solution is desired. If the external flow constants do not need to be updated, skip step 3. [If a change in step size is desired, go back to Step 2.]	To get C_f	RCL 04	
		STO 07	
		RCL 05	
		STO 08	
		RCL 03	
		A'	
		RCL 30	C_f

NOTE: If the printer is used, unlabeled output is generated in the following order: X/L, U_e/U_∞, $d(U_e/U_\infty)/d(X/L)$, θ/L, H and C_f after each increment n. Therefore, the station input in Step 4. can be placed as far forward as the external flow solution allows.

\+ Using this notation, when the X/L obtained in Step 4. is reached and the external flow constants need revision, simply reset $X/L|_{E_{i-1}}$ by RCL 03/ STO 37 and reset $U_e/U_\infty|_{i-1}$ by RCL 27/STO 35. Then finally, input the new velocity gradient.

* The program will actually proceed until at least X/L is reached and, thus, X/L should be an integer number of steps in h past the previous solution station.

b) Green's "Lag Entrainment Equation" Method

In addition to the momentum integral equation and entrainment equation, an equation for the rate of change of entrainment is used.

The momentum equation does not change and is given by:

$$\frac{d(\theta/L)}{d(X/L)} = \frac{C_f}{2} - (H + 2)\left(\frac{\theta}{L}\right)\left\{\frac{1}{(U_e/U_\infty)} \quad \frac{d(U_e/U_\infty)}{d(X/L)}\right\}$$

The entrainment equation given by:

$$\frac{dH}{d(X/L)} = \frac{1}{(\theta/L)} \cdot \frac{dH}{dH_1} \cdot \left[F - H_1 \left\{\left(\frac{\theta}{L}\right) \frac{1}{(U_e/U_\infty)} \cdot \frac{d(U_e/U_\infty)}{d(X/L)} + \frac{d(\theta/L)}{d(X/L)}\right\}\right]$$

and the equation for the entrainment parameter F is:

$$\frac{dF}{d(X/L)} = \frac{(F^2 + .02F + .2667\ C_{f_0})}{(F + .01)} \left\{ \frac{2.8}{\theta(H_1 + H)} \left[\left(.32\ C_{f_0} + .024\ F_{EQ} \right.\right.\right.$$

$$\left. + 1.2\ F^2_{EQ}\right)^{1/2} - \left(.32\ C_{f_0} + .024F + 1.2F^2\right)^{1/2}\Big]$$

$$\left. + \frac{1}{(\theta/L)} \cdot \left[\frac{\theta}{U_e} \frac{dU_e}{dX}\right]_{EQ} - \frac{1}{(U_e/U_\infty)} \frac{d(U_e/U_\infty)}{d(X/L)}\right\}$$

where a number of algebraic relations are required to complete the calculation:

i) Skin Friction*:

Start with $$C_{f_0} = \frac{.01013}{\text{Log}_{10}\ Re_\theta - 1.02} - .00075$$

Then $$H_0 = \frac{1}{1 - 6.55 \sqrt{C_{f_0}/2}}$$

And finally $$C_f = C_{f_0}\left[\frac{.9}{H/H_0 - .4} - .5\right] .$$

* $Re_\theta = U_e\theta/\nu$.

ii) Compute dH/dH_1 from:

$$\frac{dH}{dH_1} = \frac{-(H-1)^2}{1.72 + .02\,(H-1)^3}$$

iii) Find H_1 from:

$$H_1 = 3.15 + \frac{1.72}{(H-1)} - .01\,(H-1)^2$$

iv) Compute

$$\delta = \theta\,(H_1 + H)$$

v) Compute equilibrium values:

$$\left[\frac{\theta}{U}\frac{dU_e}{dX}\right]_{EQ} = \frac{1.25}{H}\left[\frac{C_f}{2} - \left(\frac{H-1}{6.432H}\right)^2\right]$$

$$F_{EQ} = H_1\left[\frac{C_f}{2} - (H+1)\left\{\frac{\theta}{U_e}\frac{dU_e}{dX}\right\}_{EQ}\right].$$

Finally, the value of F is limited to be greater than -.009. If it falls below this value, F should be reset to -.009.

The derivation of the method is contained in the original report by J. E. Green, D. J. Weeks and J. W. F. Brooman, "Prediction of Turbulent Boundary Layers and Wakes in Compressible Flow by a Lag-Entrainment Method," Royal Aircraft Establishment, TR 72231, January 1973.

USER INSTRUCTIONS -- PROGRAM 4.3(b)

STEP	ENTER	PRESS	DISPLAY
1. Initial Conditions			
i) Re_L	Re_L	STO 01	
ii) X_0/L	X_0/L	STO 03	
iii) θ/L at X_0	θ/L	STO 04	
	-	STO 07	
iv) H at X_0	H	STO 05	
v) Set F:			
a) Arbitrary F	F	STO 06	
or			
b) Use F_{EQ}	-	RCL 03	
		A'	
		RCL 33	F_{EQ}
		STO 06	
2. Select step size, h	h	STO 00	
3. Set External Flow +			
i) $X/L\|_{E_{i-1}}$	$X/L\|_{E_{i-1}}$	STO 37	
ii) $U_e/U_\infty\|_{E_{i-1}}$	U_e/U_∞	STO 35	
iii) $d(U_e/U_\infty)/d(X/L)$	$d(U_e/U_\infty)/d(X/L)$	STO 36	
4. Select station at which solution is desired, X/L*	$X/L\|_{NOM}$	A	X/L
		RCL 27	U_e/U_∞
Repeat Steps 2., 3. and 4. for each station at which the solution is desired. If the step size, n, does not need to be changed, skip Step 2. If the external flow constants do not need to be updated, skip Step 3.		RCL 04	θ/L
		RCL 05	H
		RCL 06	F
	To get C_f:	RCL 04	
		STO 07	
		RCL 05	
		STO 08	
		RCL 03	
		A'	
		RCL 30	C_f

USER INSTRUCTIONS -- PROGRAM 4.3(b) (Continued)

NOTE: If the printer is used, unlabeled output is generated in the following order: X/L, U_e/U_∞, $d(U_e/U_\infty)/d(X/L)$, θ/L, H, F and C_f after each increment n. Therefore, the station input in Step 4. can be placed as far forward as the external flow constants remain correct.

+ Using this notation, when the X/L obtained in Step 4. is reached and the external flow constants need revision, simply reset $X/L|_{E_{i-1}}$ by RCL 03/STO 37 and reset $U_e/U_\infty|_{i-1}$ by RCL 27/STO 35. Then finally, input the new velocity gradient.

* The program will actually proceed until at least X/L is reached and, thus, X/L should be an integer number of spets in h past the previous solution station in order to obtain the results at the desired station.

SAMPLE CASES:

a) The Flat Plate: U_e = $dU_e/dX = 0$ $Re_L = 10 \times 10^6$

$\frac{X_0}{L} = .20$ $\frac{\theta_0}{L} = .00008$ H = 1.46   F = .01717

Pick a step size h of .025.

STATION X/L	HEAD'S METHOD θ/L	H	C_f
.20	.0000800	1.460	.004198
.40	.0004200	1.387	.003016
.60	.0007043	1.367	.002708
.80	.0009657	1.356	.002533
.90	.0010907	1.352	.002468
1.00	.0012127	1.348	.002412

GREEN'S LAG ENTRAINMENT METHOD θ/L	H	F	C_f
.0000800	1.460	.01717	.004629
.0004346	1.350	.01365	.003115
.0007290	1.326	.01281	.002811
.0010013	1.313	.01235	.002648
.0011322	1.308	.01218	.002588
.001260	1.304	.01203	.002538

TIME

Head's -- 42 min.

Green's -- 64 min.

b) Newman's Airfoil Case*

$Re_L = 3.14 \times 10^6$ $U_\infty = 100$ ft/sec.

h = .01 to X/L = .54 h = .02 to X/L = .8 h = .0125 to X/L = .950

$\frac{X_0}{L} = .380$ $\frac{\theta_0}{L} = .001127$ H = 1.5953 F = .03537

STATION	EXTERNAL FLOW		HEAD'S METHOD			GREEN'S METHOD			
X/L	U_e/U_∞	$d(U_e/U_\infty)/d(X/L)$	θ/L	H	C_f	θ/L	H	F	C_f
.380	1.513	-1.3444	.001127	1.5953	.002037	.001127	1.5953	.03537	.001905
.400	1.486	↓	.001223	1.601	.001990	.001222	1.555	.03435	.002002
.430	1.446	↓	.001381	1.611	.001910	.001379	1.524	.032999	.002056
.460	1.405	↓	.001560	1.626	.001818	.001555	1.513	.03213	.002042
.490	1.365	↓	.001762	1.647	.001718	.001755	1.515	.03178	.001983
.510	1.338	-.8602	.001912	1.664	.001646	.001902	1.522	.03181	.001927
.600	1.261	↓	.002460	1.681	.001523	.002441	1.615	.02165	.001562
.700	1.175	↓	.003278	1.740	.001309	.003241	1.690	.03008	.001299
.800	1.089	↓	.004443	1.859	.001023	.004375	1.799	.03639	.001016
.850	1.046	↓	.005226	1.961	.000845	.005133	1.889	.03942	.000841
.900	1.003	↓	.006212	2.128+	.000628	.006081	2.027	.04263	.000632
.925	.981	↓	.006810	2.266	.006810	.006650	2.127+	.04526	.000443
.950	.960	↓	.007508	2.496	.000340	.007303	2.261	.04619	.000373

* This is essentially Example 6.10, page 196 of Cebeci and Bradshaw's Momentum Transfer in Boundary Layers, McGraw-Hill, New York, 1977. The empirical relations used in Head's Method were, in fact, developed based on the experimental data for this case (among several).

\+ Assume that separation has occurred prior to this station.

INTEGRATION OF THE ORDINARY DIFFERENTIAL EQUATIONS

The 4th order Runge-Kutta Method is used to integrate the system of ordinary differential equations (2 for Head's Method and 3 for Green's Method). The standard notation for these problems is:

$$\frac{dy_1}{dX} = f_1\,(X, y_1, y_2, y_3)$$

$$\frac{dy_2}{dX} = f_2\,(X, y_1, y_2, y_3)$$

$$\frac{dy_3}{dX} = f_3\,(X, y_1, y_2, y_3).$$

For a specified step size, h, the value of the y's at the new station $X_{i+1} = X_i + h$ are given by:

$$y_{1_{i+1}} = y_{1_i} + \frac{1}{6}\left[K_{11} + 2K_{12} + 2K_{13} + K_{14}\right]$$

$$y_{2_{i+1}} = y_{2_i} + \frac{1}{6}\left[K_{21} + 2K_{22} + 2K_{23} + K_{24}\right]$$

$$y_{3_{i+1}} = y_{3_i} + \frac{1}{6}\left[K_{31} + 2K_{32} + 2K_{33} + K_{34}\right]$$

where

$$K_{11} = h\, f_1\left(X_i, y_{1_i}, y_{2_i}, y_{3_i}\right)$$

$$K_{21} = h\, f_2\left(X_i, y_{1_i}, y_{2_i}, y_{3_i}\right)$$

$$K_{31} = h\, f_3\left(X_i, y_{1_i}, y_{2_i}, y_{3_i}\right)$$

$$K_{12} = h\, f_1\left(X_i + h/2, y_{1_i} + K_{11}/2, y_{2_i} + k_{21}/2, y_{3_i} + K_{31}/2\right)$$

$$K_{22} = h\, f_2\left(X_i + h/2, y_{1_i} + K_{11}/2, y_{2_i} + K_{21}/2, y_{3_i} + K_{31}/2\right)$$

$$K_{32} = h\, f_3\left(X_i + h/2, y_{1_i} + K_{11}/2, y_{2_i} + K_{21}/2, y_{3_i} + K_{31}/2\right)$$

$$K_{13} = h\, f_1\left(X_i + h/2, y_{1_i} + K_{12}/2, y_{2_i} + K_{22}/2, y_{3_i} + K_{32}/2\right)$$

$$K_{23} = h\, f_2\left(X_i + h/2, y_{1_i} + K_{12}/2, y_{2_i} + K_{22}/2, y_{3_i} + K_{32}/2\right)$$

$$K_{33} = h\, f_3\left(X_i + h/2, y_{1_i} + K_{12}/2, y_{2_i} + K_{22}/2, y_{3_i} + K_{32}/2\right)$$

$$K_{14} = h\, f_1\left(X_i + h, y_{1_i} + K_{13}, y_{2_i} + K_{23}, y_{3_i} + K_{33}\right)$$

$$K_{24} = h\, f_2\left(X_i + h, y_{1_i} + K_{13}, y_{2_i} + K_{23}, y_{3_i} + K_{33}\right)$$

$$K_{34} = h\, f_3\left(X_i + h, y_{1_i} + K_{13}, y_{2_i} + K_{23}, y_{3_i} + K_{33}\right)$$

If only two equations are solved, K_{31}, K_{32}, K_{33}, K_{34} and f_3 are not needed.

Note that the 3 equation rules given here can also be used to integrate the Faulkner-Skan equations, or any other system of ordinary differential equations.

The step size selection depends on the rate of change of the external flow. The error of this method is about $O(h^5)$ and for cases with slowly changing freestreams, h = .05 is probably safe. Near the trailing edge, the user will often want to get the solution at every .01 and, thus h = .01, will probably be chosen even though the numerical solution does not require this step size. Step size selection is an area in which the user can gain some experience by experimenting with the step size effects on the solution.

DETAILS OF PROGRAM 4.3(a)

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-3	Routine A, start program and save X_{FINAL}.	0	h
		1	R_L
4-21	Routine B, start Runge-Kutta solution by setting y_i's for f_i evaluation for 1st call to function, store K_{i1}.	2	X_{FINAL}
		3	$X_{START/FINISH}$
22-36	2nd function evaluation, store K_{i2}.	4	θ initial and final
		5	H initial and final
37-51	3rd function evaluation, store K_{i3}.	6	-
52-65	4th function evaluation, store K_{i4}.	7	θ intermediate
		8	H intermediate
66-112	Loop to compute new values of θ and H.	9	-
113-124	Move X_{START} forward by h.	10	K_{11}
125-142	Print out results if printer is used.	11	K_{12}
143-148	Test to see if X_{FINAL} has been reached.	12	K_{13}
149-177	Routine E', store K_{ij}'s.	13	K_{14}
178-219	Routine C', set arguments for function evaluation.	14	K_{21}
		15	K_{22}
220-233	Routine A', provides f_i, starts by computing and saving Re_θ.	16	K_{23}
234-264	Compute and store C_f.	17	K_{24}
265-284	Compute and store $d\theta/dX$.		
285-322	Compute and store H_1.	18-21	-
323-351	Compute and save dH/dH_1.	22	$d\theta/dX$

DETAILS OF PROGRAM 4.3(a) Continued

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
352-388	Compute and store H_1.	23	dH/dX
389-419	Compute and store dH_1/dH.	24	-
420-448	Compute and store F.	25	$\overline{X}$
449-476	Compute and store dH/dX.	26	U_e'/U_e
477-506	Routine E, inviscid flow routine for piecewise linear edge velocity.	27	U_e
		28	Re_{θ}
	INVISCID FLOW ROUTINE	29	dH/dH_1
	Label E, X/L is in display on call, U_e/U_∞ should be stored in R_{27} and placed in display on return; $d(U_e/U_\infty)/d(X/L)/(U_e/U_\infty)$ should be stored in R_{26}.	30	C_f
		31	H_1
		32	-
		33	F
		34	-
	External Flow	35	$U_e/U_\infty\|_{E_{i-1}}$
	External Flow	36	dU_e/dX
	External Flow	37	$X/L\|_{E_{i-1}}$
	External Flow	38	-
	External Flow	39	-

DETAILS OF PROGRAM 4.3(b)

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-3	Routine A, start program and save X_{FINAL}.	0	h
4-21	Routine B, start Runge-Kutta solution by setting y_i's for f_i evaluation for 1st call to function, store K_{i1}.	1	R_L
22-36	2nd function evaluation, store K_{i2}.	2	X_{FINAL}
37-51	3rd function evaluation, store K_{i3}.	3	X_{START}
52-65	4th function evaluation, store K_{i4}.	4	θ initial and final
66-99	Loop to compute new values of θ, H and F.	5	H initial and final
100-114	Check that $F \geq -.009$ and set if required.	6	F initial and final
115-126	Move X_{START} forward by h.	7	θ intermediate
127-147	Print out results if printer is used.	8	H intermediate
148-153	Test to see if X_{FINAL} has been reached.	9	F intermediate
154-160	Routine D', register manipulation for summation.	10	K_{11}
161-189	Routine E', store K_{ij}'s.	11	K_{12}
190-246	Routine C', set arguments for function evaluation.	12	K_{13}
247-260	Routine A', provides f_i, starts by computing and saving Re_θ.	13	K_{14}
261-288	Compute and store C_{f_0}.	14	K_{21}
289-329	Compute and store C_f.	15	K_{22}
330-349	Compute and store $d\theta/dX$.	16	K_{23}
		17	K_{24}
		18	K_{31}
		19	K_{32}
		20	K_{33}
		21	K_{34}

DETAILS OF PROGRAM 4.3(b) Continued)

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
350-421	Compute and store dH/dX.	22	$d\theta/dX$
422-452	Compute and store $[\theta/U_e \cdot dU_e/dX]_{EQ}$.	23	dH/dX
		24	dF/dX
453-475	Compute and store F_{EQ}.	25	$\overline{X}$
476-592	Compute and store dF/dX.	26	U_e'/U_e
593-622	Routine E, inviscid flow routine for piecewise linear edge velocity.	27	U_e
		28	Re_θ
INVISCID FLOW ROUTINE		29	C_{f_0}
Label E, X/L is in display on call, U_e/U_∞ should be stored in R_{27} and placed in display on return; $d(U_e/U_\infty)/d(X/L)/(U_e/U_\infty)$ should be stored in R_{26}.		30	C_f
		31	H_1
		32	$[\theta/U_e \cdot dU_e/dX]_{EQ}$
		33	F_{EQ}
		34	δ/L
	External Flow	35	$U_e/U_\infty\|_{E_{i-1}}$
	External Flow	36	dU_e/dX
	External Flow	37	$X/L\|_{E_{i-1}}$
	External Flow	38	-
	External Flow	39	-

4.4 COMPRESSIBLE FLAT PLATE SKIN FRICTION FORMULAS

These programs provide estimates of the skin friction on surfaces with no pressure gradients, including both Mach number and wall temperature effects. Although these estimates are crude compared to the previous methods, they form the basis for most skin friction estimating techniques used in the aircraft industry. (Actual estimates include a number of additional factors. As an example of the use of friction estimates in aircraft analysis, see the XB-70 analysis contained in NASA TP 1515, February 1980.)

a) Laminar Flow

The approach used is known as the Eckert Reference Temperature Method and this particular version is the one given by F. M. White in Viscous Fluid Flow, McGraw-Hill, New York, 1974, pp. 589-590. In this method, the incompressible formula is used, with the fluid properties chosen at a specified reference temperature, which includes both Mach number and wall temperature effects.

METHOD

Given fluid properties:

Prandtl number, Pr

Recovery factor, $r = (\sqrt{Pr})$

Specific heat ratio, γ

Edge temperature, Te (°R)

Then for a given Mach number, M_e, and ratio of wall temperature to adiabatic wall temperature T_W/T_{AW}; compute:

$$\frac{T_W}{T_e} = \frac{T_W}{T_{AW}} \cdot \left(1 + r \cdot \frac{\gamma-1}{2} \cdot M_e^2\right) .$$

[Remember that $T_{AW} = T_e \left(1 + r \frac{\gamma-1}{2} M_e^2\right)$] and then compute the reference temperature:

$$\frac{T^*}{T_e} \cong .5 + .039\, M_e^2 + .5\left(\frac{T_W}{T_e}\right) .$$

The Chapman-Rubesin constant based on the reference temperature and Sutherland's viscosity law is then computed from:

$$C^* = \left(\frac{T^*}{T_e}\right)^{1/2}\left(\frac{1 + K/T_e}{T^*/T_e + K/T_e}\right) \quad \text{where } K = 200°R \text{ for air.}$$

Finally, the local friction coefficient (τ_W/q) is found from the standard Blasius formula, with C* added; i.e.,

$$C_f = \frac{.664\sqrt{C^*}}{\sqrt{Re_X}}$$

and

$$C_F = 2C_f$$

where

$$C_F = \frac{F}{qX} = \frac{1}{X}\int_0^X C_f(X')\,dX'.$$

Recall that C_F accounts for one side of the plate only, so that if both sides are required for a drag estimate, then the friction drag is twice C_F.

Note that the results are not sensitive to the value of edge temperature for low Mach numbers and, therefore, an exact specification of T_e is not required.

USER INSTRUCTIONS -- PROGRAM 4.4(a)

STEP	ENTER	PRESS	DISPLAY
1. Enter Mach number	M_e	A	390 (default $T_e \sim$°R)
Step 2. is required only if the default values $Pr = .72$ $r = \sqrt{Pr}$ $\gamma = 1.4$ $T_e = 390$°R require change.			
2. Reset property values as required			
i) Change Pr ($r = \sqrt{Pr}$)	Pr	A'	$r = \sqrt{Pr}$
ii) Change γ	γ	B'	γ
iii) Change T_e	T_e(°R)	C'	T_e
iv) Change r	r	D'	r
3. Input wall temperature ratio, T_W/T_{AW}	T_W/T_{AW}	B	T_W/T_{AW}
4. Enter Reynolds number based on length, X NOTE: Include exponent.	Re_X	C RCL 10	C_F C_f

SAMPLE CASES: $Pr = .72$, $r = \sqrt{Pr}$, $\gamma = 1.4$, $T_e = 390$°R

$T_W/T_{AW} = 1$, $Re_X = 1.X10^6$

M_e	C_F
0	.001328
.6	.001323
1.2	.001309
2.0	.001275
4.0	.001148
10.0	.000838

$M_e = 2.0$, $Re_X = 1.X10^6$

T_W/T_{AW}	C_F
.10	.001353
.60	.001311
1.00	.001275
2.00	.001196

DETAILS OF PROGRAM 4.4(a)

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-22	Routine A. Read M, set default property values	0	M_e
		1	Pr
23-30	Routine A', read Pr, calculate r	2	γ
31-35	Routine B', read γ	3	T_e
36-40	Routine C', read T_e	4	T_W/T_{AW}
41-45	Routine D', read r	5	r
46-50	Routine B, read T_W/T_{AW}	6	Re_X
51-80	Routine C, read Re_X, Calculate T_W/T_e	7	T_W/T_e
		8	T^*/T_e
81-99	Calculate T^*/T_e	9	C*
100-128	Calculate C*	10	C_f
129-143	Calculate C_f	11	C_F
144-151	Calculate C_F		

b) Turbulent Flow

For turbulent flow, the so-called van Driest II Method is employed. This method was selected based on the recommendation of E. J. Hopkins and M. Inouye, contained in "An Evaluation of Theories for Predicting Turbulent Skin Friction and Heat Transfer on Flat Plates at Supersonic and Hypersonic Mach Numbers," AIAA J., Vol. 9, No. 6, June 1971, pp. 993-1003. The particular algorithm is taken from NASA TN D-6945, "Charts for Predicting Turbulent Skin Friction From the van Driest Method (II)," also by E. J. Hopkins and dated October 1972.

METHOD

Supply the following information:

i) The ratio of specific heats, γ ($\sim$ 1.4)

ii) The turbulent recovery factor, r ($\sim$ .88)

iii) The edge temperature, T_e ($\sim$ 222°K).

Then, for a given Mach number, M_e, and ratio of wall temperature to adiabatic wall temperature, T_W/T_{AW}, the calculation is started by computing the following constants:

$$m = \frac{\gamma - 1}{2} M_e^2$$

$$F = \frac{T_W}{T_e} = \frac{T_W}{T_{AW}} \cdot \frac{T_{AW}}{T_e}$$

where

$$\frac{T_{AW}}{T_e} = 1 + rm$$

$$T_W = F \cdot T_e$$

$$A = \left(\frac{rm}{F}\right)^{1/2}$$

$$B = \frac{1 + rm - F}{F}$$

$$\alpha = \frac{2A^2 - B}{(4A^2 + B^2)^{1/2}}$$

$$\beta = \frac{B}{(4A^2 + B^2)^{1/2}}$$

$$F_c = \frac{rm}{(\sin^{-1}\alpha + \sin^{-1}\beta)^2} \qquad M_e > 0.1$$

$$= \left[\frac{1 + \sqrt{F}}{2}\right]^2 \qquad M_e \leq 0.1$$

$$F_\theta = \frac{\mu_e}{\mu_W} = \sqrt{\frac{1}{F}} \left[\frac{1 + \frac{122}{T_W} \times 10^{-5/T_W}}{1 + \frac{122}{T_e} \times 10^{-5/T_e}}\right]$$

(Keyes viscosity law)

and finally,

$$F_X = F_\theta / F_c.$$

In the following analysis, barred quantities denote "incompressible" variables and are actually only intermediate variables, not to be used except in obtaining the final results.

Case 1: Given Re_θ

Using the constants computed previously, continue the sequence as follows:

$$\overline{Re}_\theta = F_\theta \, Re_\theta$$

$$\overline{C}_F = \left[\frac{.242}{\text{Log}_{10}\,(2\cdot\overline{Re}_\theta)}\right]^2$$

(This is a form of the Karman-Schoenherr formula.)

$$\overline{C}_f = \frac{.242\,\overline{C}_F}{.242 + .8686\sqrt{\overline{C}_F}}$$

$$Re_X = \frac{2\cdot\overline{Re}_\theta}{\overline{C}_F\cdot F_X}$$

finally,

$$C_f = \frac{\overline{C}_f}{F_c}$$

and

$$C_F = \frac{\overline{C}_F}{F_c}\,.$$

Case 2: Given Re_X

For this case, an iteration is required. Fortunately, Newton's method converges reliably. Thus, the calculation proceeds as follows:

$$\overline{Re}_X = F_X \cdot Re_X$$

now solve

$$\frac{.242}{\sqrt{\overline{C}_F}} = \text{Log}_{10}\,(\overline{Re}_X\,\overline{C}_F)$$

for $\overline{C}_F$.

Use as an initial guess:

$$\overline{C}_F{}^\circ = \frac{.074}{\overline{Re}_X{}^{.20}}$$

Then, Newton's method applied to this problem; i.e.:

$$f\ (\overline{C}_F) = 0 \rightarrow \overline{C}_F^{\ i+1} = \overline{C}_F^{\ i} - f/f'$$

becomes for this equation:

$$\overline{C}_F^{\ i+1} = \overline{C}_F^{\ i} \left[1 + \frac{\left\{.242 - \sqrt{\overline{C}_F^{\ i}} \cdot \mathrm{Log}_{10}\ (\overline{Re}_X\ \overline{C}_F^{\ i})\right\}}{\left\{.121 + \sqrt{\overline{C}_F^{\ i}}/\ell n\ 10\right\}} \right].$$

Once this iteration is completed and $\overline{C}_F$ is known, $\overline{C}_f$ is computed from:

$$\overline{C}_f = \frac{.242\ \overline{C}_F}{.242 + .8686\ \sqrt{\overline{C}_F}}$$

and

$$Re_\theta = \frac{\overline{C}_F\ \overline{Re}_X}{2F_\theta}.$$

The final results are thus:

$$C_f = \frac{\overline{C}_f}{F_c}$$

and

$$C_F = \frac{\overline{C}_F}{F_c}.$$

Note that this value applies to one side of a plate only, so it must be doubled if the friction on both sides is desired.

In this formulation, the choice of edge temperature is not crucial at low Mach numbers and, hence, the default value should be adequate in most cases.

USER INSTRUCTIONS -- PROGRAM 4.4(b)

STEP	ENTER	PRESS	DISPLAY
1. Enter the Mach number. Step 2. is required only if the default values $r = .88$, $\gamma = 1.40$, $T_e = 222°K$ require change.	M_e	A	222. (T_e)
2. Reset property values as required.			
i) Change r	r	A'	r
ii) Change γ	γ	B'	γ
iii) Change T_e	T_e	C'	T_e
3. Input wall temperature ratio, T_W/T_{AW}	T_W/T_{AW}	B	F_X
Enter either Re_θ or Re_X.			
4a) Enter Reynolds number based on momentum thickness	Re_θ	C RCL 25	C_F C_f
OR			
4b) Enter Reynolds number based on length, X. (Note: Include exponent.)	Re_X	D RCL 25	C_F C_f

SAMPLE CASES: $r = .88$, $\gamma = 1.40$, $T_e = 222°K$

a) Re_θ given: $Re_\theta = 1.0 \times 10^5$

$T_W/T_{AW} = 1.0$

M_e	C_F	C_f
0	.00208	.00179
2	.00155	.00132
4	.00093	.00079
6	.00059	.00050

$M_e = 4$

T_W/T_{AW}	C_F	C_f
.2	.00152	.00131
.6	.00114	.00097
1.0	.00093	.00079

b) Re_X given $Re_X = 20 \times 10^6$

$T_W/T_{AW} = 1.0$

M_e	C_F	C_f
0	.00263	.00222
2	.00206	.00172
4	.00137	.00113
6	.00096	.00078

$M_e = 4$

T_W/T_{AW}	C_F	C_f
.2	.00201	.00169
.6	.00161	.00133
1.0	.00137	.00113

DETAILS OF PROGRAM 4.4(b)

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-19	Routine A, store M_e, set default values of parameters.	0	M_e
		1	T_W/T_{AW}
		2	r
20-24	Routine A', read r.	3	γ
25-29	Routine B', read γ.	4	T_e
30-34	Routine C', read T_e.	5	$T_W/T_e = f$
35-55	Routine B, store T_W/T_{AW}, compute m.	6	m
		7	A
56-77	Compute rm and F.	8	B
78-85	Compute T_W	9	α
86-96	Compute A	10	β
97-112	Compute B	11	F_c
113-128	Compute $\sqrt{4A^2 - B^2}$	12	F_θ
129-145	Compute α	13	F_X
146-154	Compute β	14	$(4A^2 + B^2)^{1/2}$
155-201	Compute F_c	15	rm
202-253	Compute F_θ	16	T_W
254-262	Compute F_X	17/18/19	-
263-274	Routine C, store Re_θ, calculate Re_θ	20	Re_θ
		21	$\overline{Re}_\theta$
275-291	Compute $\overline{C}_F$	22	$\overline{C}_F$
292-293	Call Routine SUM to calculate $\overline{C}_f$	23	$\overline{C}_f$
294-307	Calculate Re_X	24	Re_X
308-309	Skip Re_X option steps	25	C_f
		26	C_F
		27	$\overline{Re}_X$

DETAILS OF PROGRAM 4.4(b) (Continued)

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
310-321	Routine D, store Re_X, $\overline{Re}_X$		
322-346	Calculate $\overline{C}_F°$ and set up iteration		
347-407	Iteration loop for $\overline{C}_F$		
408-409	Call Routine SUM to calculate $\overline{C}_f$		
410-421	Calculate Re_θ		
422-444	Calculate C_f and C_F		
445-475	Routine SUM, computes $\overline{C}_f$		

4.5 TRANSITION POSITION EFFECTS ON SKIN FRICTION

In many cases, the boundary layer is initially laminar and then becomes turbulent at some point downstream. Several formulas are available to estimate the drag of a flat plate when this occurs. This program computes the estimated skin friction from three of the methods given the Reynolds number and transition location. For cases in which the boundary layer is tripped deliberately by attaching grit to the model, these formulas do not include the drag due to the boundary layer trip (C_F is for one side of plate only).

a) Liu's Formula: C. Liu, "Drag of a Flat Plate with Transition in the Absence of Pressure Gradient," Journal of Aircraft, Vol. 9, No. 7, July 1972, pp. 509-510.

Give i) Re_L, the Reynolds number based on the plate length

ii) Xc/L, the transition location

then compute:

$$Re_C = \left(\frac{Xc}{L}\right) Re_L$$

$$Re_{ci} = \left[18.44\ Re_c^{1/2}\right]^{5/4}$$

$$\left(\frac{L-Xi}{L}\right) = \left[\frac{Re_{ci}}{Re_L} + 1 - \frac{Xc}{L}\right]$$

$$Re_{L-X_i} = Re_L \left(\frac{L-X_i}{L}\right)$$

then:

$$C_F = \frac{1.328}{\sqrt{Re_L}} \cdot \left(\frac{Xc}{L}\right)^{1/2} + .455\left\{\left(\frac{L-X_i}{L}\right)\frac{1}{\left(\text{Log } Re_{L-X_i}\right)^{2.58}} - \frac{Re_{ci}}{Re_L} \cdot \frac{1}{\left(\text{Log } Re_{ci}\right)^{2.58}}\right\}$$

NOTE: Log refers to base 10.

b) Collar's Formula: A. R. Collar, "A Closed Formula For the Drag of a Flat Plate with Transition in the Absence of a Pressure Gradient," J. Of the Royal Aeronautical Society, Vol. 64, Jan. 1960, pp. 38-39.

Given Xc/L and Re_L, compute $Re_c = \left(\frac{Xc}{L}\right) Re_L$

and

$$C_F = \frac{0.074}{Re_L} \left[Re_L - Re_c + 36.9\ Re_c^{5/8} \right]^{4/5}$$

c) Schlicting's Formula: See T. Cebeci and P. Bradshaw, Momentum Transfer in Boundary Layers, McGraw-Hill, New York, 1977, pp. 187.

Given Xc/L and Re_L, compute $Re_c = \left(\frac{Xc}{L}\right) Re_L$

and

$$C_F = \frac{.455}{\left[\text{Log } Re_L\right]^{2.58}} - \left(\frac{Xc}{L}\right)\left[\frac{.455}{\left\{\text{Log } Re_c\right\}^{2.58}} - \frac{1.328}{\sqrt{Re_c}}\right]$$

Note that C_F is the friction drag coefficient for one side of a plate and, if the drag is desired including both sides, C_F must be doubled.

USER INSTRUCTIONS -- PROGRAM 4.5

STEP	ENTER	PRESS	DISPLAY
1. Enter Reynolds number	Re_L	A	Re_L
2. Enter transition location	Xc/L	B	Xc/L
3. Start computation	-	C	C_F (Liu)
4. Use Collar's Formula	-	D	C_F (Collar)
5. Use Schlicting's Formula	-	E	C_F (Schlicting)

SAMPLE CASE: $Re_L = 1.0 \times 10^6$

	$C_F \times 10^4$		
Xc/L	Liu	Collar	Schlicting
.001	44.7	46.8	44.9
.01	44.7	46.8	44.8
.05	44.0	46.0	43.5
.10	42.9	44.8	41.8
.30	37.4	39.0	35.0
.50	31.2	32.4	28.6
.70	24.6	25.3	22.3
.90	17.3	17.5	16.3
.95	15.3	15.4	14.8
1.00	13.3	13.3	13.3

DETAILS OF PROGRAM 4.5

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-4	Routine A, store Re_L	0	Re_L
5-9	Routine B, store Xc/L	1	Xc/L
10-20	Routine C, compute Re_c	2	Re_c
21-39	Compute Re_{ci}	3	Re_{ci}
40-52	Compute $(L-X_i)/L$	4	Re_{L-X_i}
53-60	Compute Re_{L-X_i}	5	C_F (Liu)
61-121	Liu's C_F Formula	6	$\left(\frac{L-X_i}{L}\right)$
122-161	Collar's C_F Formula	7	C_F (Collar)
162-214	Schlicting's C_F Formula		

4.6 PROFILE DRAG -- THE SQUIRE-YOUNG FORMULA

In addition to skin friction drag, bodies also experience a drag due to the pressure distribution. This drag is usually called form drag, but is also known as pressure drag. The sum of the friction and form drag is known as the profile drag. This is the drag value which is usually sought. The Squire-Young Formula provides a means of predicting the profile drag by relating the momentum defect far downstream to the values of the flowfield at the airfoil trailing edge. This program evaluates the Squire-Young Formula given the trailing edge information and provides the value of the profile drag. A derivation and discussion of this formula can be found in Incompressible Aerodynamics, edited by B. Thwaites, Oxford, 1960, pp. 179-181.

Given the momentum thickness, θ/c, shape factor, H, and velocity, U_e/U_∞, at the trailing edge on the upper and lower surface, the drag can be found from:

$$C_D = \left[2\left(\frac{\theta}{c}\right)_{TE} \cdot \left(\frac{U_e}{U_\infty}\right)_{TE}^{\frac{H_{TE}+5}{2}}\right]_{UP} + \left[2\left(\frac{\theta}{c}\right)_{TE} \cdot \left(\frac{U_e}{U_\infty}\right)_{TE}^{\frac{H_{TE}+5}{2}}\right]_{LOW}$$

When C_p is given at the trailing edge, U_e/U_∞ is found from:

$$\left(\frac{U_e}{U_\infty}\right) = \sqrt{1 - C_p} \quad .$$

Finally, note that drag is usually described in drag counts rather than as a drag coefficient.

1 drag count = .0001 in C_D.

USER INSTRUCTIONS -- PROGRAM 4.6

STEP	ENTER	PRESS	DISPLAY
1. Input upper surface information:			
a) $(\theta/c)_u$	θ/c	A	
b) H_{TE_u}	H_{TE}	B	
c1) $[U_e/U_\infty]_u$	U_e/U_∞	C	
OR			
c2) C_{p_u}	C_{p_u}	D	
2. If lower surface is different:			
a) $(\theta/c)_\ell$	θ/c	A'	
b) H_{TE_ℓ}	H_{TE}	B'	
c1) $[U_e/U_\infty]_\ell$	U_e/U_∞	C'	
OR			
c2) C_{p_ℓ}	C_{p_ℓ}	D'	
3. Compute the drag coefficient	-	E	C_D

SAMPLE CASE: For purposes of testing the program coding, assume the following values:

1. Laminar Flat Plate: $\theta/c = .000664$, $H_{TE} = 2.59$, $U_e/U_\infty = 1$

   $C_D = 26.56$ cts.

2. Symmetric: $\theta/c = 1.5 \cdot \theta/c|_{FLT.PLT.}$, $H_{TE} = 3.0$, $C_p = .15$

   $C_D = 28.78$ cts.

3. Asymmetric: $\theta/c|_u = 1.50, \theta/c|_{FLT.PLT.}$, $H_{TE_u} = 3.4$, $C_{p_u} = .10$

   $\theta/c|_\ell = 1.25, \theta/c|_{FLT.PLT.}$, $H_{TE_\ell} = 3.1$, $C_{p_\ell} = .15$

   $C_D = 27.91$ cts.

DETAILS OF PROGRAM 4.6

STEPS	PROGRAM DESCRIPTION	R	REGISTER CONTENTS
0-6	Routine A, store θ/c in u and ℓ registers	0	C_D
		1	$\theta/c\|_u$
7-13	Routine B, store H_{TE} in u and ℓ registers	2	$\theta/c\|_\ell$
14-20	Routine C, store U_e/U_∞ in u and ℓ registers	3	H_{TE_u}
		4	H_{TE_ℓ}
21-40	Routine D, store C_p and U_e/U_∞ in u and ℓ registers	5	$U_e/U_{\infty u}$
41-45	Routine A', store θ/c in ℓ	6	$U_e/U_{\infty\ell}$
46-50	Routine B', store H_{TE} in ℓ	7	C_{p_u}
51-55	Routine C', store U_e/U_∞ in ℓ	8	$C_{p\ell}$
56-71	Routine D', store C_p and U_e/U_∞ in ℓ	9	C_{D_u}
72-98	Routine E, start calculation, compute C_{D_u}	10	C_{D_ℓ}
99-127	Calculate C_{D_ℓ} and add to C_{D_u}		