LEVEL

COMPUTATION METHODS

R.G. LAMBERT

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COMPUTATION METHODS

LIST OF FIGURES, TABLES AND COMPUTER PROGRAM LISTINGS

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COMPUTATION METHODS
BASIC FUNCTIONS

INTRODUCTION

The computation methods described in this section are for use in fatigue analyses. The functions to be computed are the Gamma Function, the two Incomplete Gamma Functions, an Error Function and its Inverse, a Probability of Failure Function, and a transcendental accelerated test level function.

These methods are intended to be user oriented. The user is given a choice of methods. Tables and interpolation methods are included for those who prefer to use tables. Techniques are also included for use with calculators, with the modern TI-59, HP-67, HP-34C and HP-41C Programmable Calculators and with large computers (i.e. Basic Language). Both numerical integration and closed form equation methods are given. Examples using each method are worked out. Computer program listings are also shown.

It should be noted that the same programs can be used for both the HP-67 and HP-41C programmable calculators. Only the HP-67 is referred to in the listings for simplicity.

A. GAMMA FUNCTION COMPUTATION

Definition:

The Gamma Function is defined as follows:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$\Gamma(\alpha)$ is undefined for $\alpha = 0$ and for negative integer values of α. For fatigue analyses $\alpha > 1$. Figure 1 shows $\Gamma(\alpha)$ versus α for $0 < \alpha < 4$. The curve increases monotonically for $\alpha > 4$.

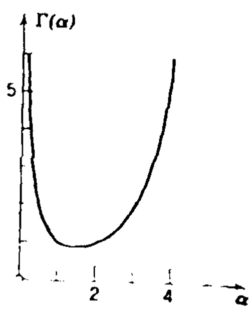


Figure 1 Plot of Gamma Function Versus α

TABULAR METHOD

Table I (Table 6.1 pages 267-270 of Abramowitz [1]) gives values to ten places of $\Gamma(\alpha)$ versus α for $1 \leq \alpha \leq 2$. Table II (Table 6.3 pages 272, 273 of Abramowitz [1]) gives values to eleven places of $\Gamma(\alpha)$ for integer and half-integer values of α for $1 \leq \alpha \leq 101$.

EXAMPLE

FIND: $\Gamma(1.225)$

SOLUTION: In Table I (Table 6.1 page 267 [1])

$$\Gamma(1.225) = 0.9119156071$$

EXAMPLE:

FIND: $\Gamma(8.5)$

SOLUTION: In Table II (Table 6.3 page 272 [1])

$$\Gamma(8.5) = 1.4034407 \times 10^4$$

The following expression can be used recursively:

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$

EXAMPLE:

FIND: $\Gamma(5.64)$

SOLUTION: $\Gamma(5.64) = 4.64 \times 3.64 \times 2.64 \times 1.64 \times \Gamma(1.64)$

In Table 6.1 page 269 [1]

$$\Gamma(1.64) = 0.8986420302$$

$$\therefore \quad \Gamma(5.64) = 65.71338911$$

Table I GAMMA FUNCTION ($1 \leq \alpha \leq 2$)

GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS Table 6.1

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.000	1.00000 00000	0.00000 00000	-0.57721 56649	1.64493 40668	0.000
1.005	0.99713 85354	-0.00286 55666	-0.56902 09113	1.63299 41567	0.005
1.010	0.99432 58512	-0.00569 03079	-0.56088 54579	1.62121 35283	0.010
1.015	0.99156 12888	-0.00847 45187	-0.55280 85156	1.60958 91824	0.015
1.020	0.98884 42033	-0.01121 84893	-0.54478 93105	1.59811 81919	0.020
1.025	0.98617 39633	-0.01392 25067	-0.53682 70828	1.58679 76993	0.025
1.030	0.98354 99506	-0.01658 68539	-0.52892 10873	1.57562 49154	0.030
1.035	0.98097 15606	-0.01921 18101	-0.52107 05921	1.56459 71163	0.035
1.040	0.97843 82009	-0.02179 76511	-0.51327 48789	1.55371 16426	0.040
1.045	0.97594 92919	-0.02434 46490	-0.50553 32428	1.54296 58968	0.045
1.050	0.97350 42656	-0.02685 30725	-0.49784 49913	1.53235 73421	0.050
1.055	0.97110 25663	-0.02932 31868	-0.49020 94448	1.52188 35001	0.055
1.060	0.96874 36495	-0.03175 52537	-0.48262 59358	1.51154 19500	0.060
1.065	0.96642 69823	-0.03414 95318	-0.47509 38088	1.50133 03259	0.065
1.070	0.96415 20425	-0.03650 62763	-0.46761 24199	1.49124 63164	0.070
1.075	0.96191 83189	-0.03882 57395	-0.46018 11367	1.48128 76622	0.075
1.080	0.95972 53107	-0.04110 81702	-0.45279 93380	1.47145 21556	0.080
1.085	0.95757 25273	-0.04335 38243	-0.44546 64135	1.46173 76377	0.085
1.090	0.95545 94882	-0.04556 29148	-0.43818 17635	1.45214 19988	0.090
1.095	0.95338 57227	-0.04773 57114	-0.43094 47988	1.44266 31755	0.095
1.100	0.95135 07699	-0.04987 24413	-0.42375 49404	1.43329 91508	0.100
1.105	0.94935 41778	-0.05197 33384	-0.41661 16193	1.42404 79514	0.105
1.110	0.94739 55040	-0.05403 86341	-0.40951 42761	1.41490 76482	0.110
1.115	0.94547 43149	-0.05606 85568	-0.40246 23611	1.40587 63535	0.115
1.120	0.94359 01856	-0.05806 33325	-0.39545 53339	1.39695 22213	0.120
1.125	0.94174 26997	-0.06002 31841	-0.38849 26633	1.38813 34449	0.125
1.130	0.93993 14497	-0.06194 83322	-0.38157 38268	1.37941 82573	0.130
1.135	0.93815 60356	-0.06383 89946	-0.37469 83110	1.37080 49288	0.135
1.140	0.93641 60657	-0.06569 53867	-0.36786 56106	1.36229 17670	0.140
1.145	0.93471 11562	-0.06751 77212	-0.36107 52291	1.35387 71152	0.145
1.150	0.93304 09311	-0.06930 62087	-0.35432 66780	1.34555 93520	0.150
1.155	0.93140 50217	-0.07106 10569	-0.34761 94768	1.33733 68900	0.155
1.160	0.92980 30666	-0.07278 24716	-0.34095 31528	1.32920 81752	0.160
1.165	0.92823 47120	-0.07447 06558	-0.33432 72413	1.32117 16859	0.165
1.170	0.92669 96106	-0.07612 58106	-0.32774 12847	1.31322 59322	0.170
1.175	0.92519 74225	-0.07774 81345	-0.32119 48332	1.30536 94548	0.175
1.180	0.92372 78143	-0.07933 78240	-0.31468 74438	1.29760 08248	0.180
1.185	0.92229 04591	-0.08089 50733	-0.30821 86809	1.28991 86421	0.185
1.190	0.92088 50371	-0.08242 00745	-0.30178 81156	1.28232 15358	0.190
1.195	0.91951 12341	-0.08391 30174	-0.29539 53259	1.27480 81622	0.195
1.200	0.91816 87424	-0.08537 40900	-0.28903 98966	1.26737 72054	0.200
1.205	0.91685 72606	-0.08680 34780	-0.28272 14187	1.26002 73755	0.205
1.210	0.91557 64930	-0.08820 13651	-0.27643 94897	1.25275 74090	0.210
1.215	0.91432 61500	-0.08956 79331	-0.27019 37135	1.24556 60671	0.215
1.220	0.91310 59475	-0.09090 33619	-0.26398 37000	1.23845 21360	0.220
1.225	0.91191 56071	-0.09220 78291	-0.25780 90652	1.23141 44258	0.225
1.230	0.91075 48564	-0.09348 15108	-0.25166 94307	1.22445 17702	0.230
1.235	0.90962 34274	-0.09472 45811	-0.24556 44243	1.21756 30254	0.235
1.240	0.90852 10583	-0.09593 72122	-0.23949 36791	1.21074 70707	0.240
1.245	0.90744 74922	-0.09711 95744	-0.23345 68341	1.20400 28063	0.245
1.250	0.90640 24771	-0.09827 18364	-0.22745 35334	1.19732 91545	0.250
	$y!$	$\ln y!$	$\frac{d}{dy} \ln y!$	$\frac{d^2}{dy^2} \ln y!$	y
	$\left[\begin{smallmatrix}(-6)6\\5\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)5\\5\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-6)7\\5\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-5)2\\5\end{smallmatrix}\right]$	

For $x>2$ see Examples 1–4. $\log_{10} e = 0.43429\ 44819$

Compiled from H. T. Davis, Tables of the higher mathematical functions, 2 vols. (Principia Press, Bloomington, Ind., 1933, 1935) (with permission). Known error has been corrected.

Table I (Cont'd)

Table 6.1 GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.250	0.90640 24771	-0.09827 18364	-0.22745 35334	1.19732 91545	0.250
1.255	0.90538 57663	-0.09939 41651	-0.22148 34266	1.19072 50579	0.255
1.260	0.90439 71178	-0.10048 67254	-0.21554 61686	1.18418 94799	0.260
1.265	0.90343 62946	-0.10154 96809	-0.20964 14193	1.17772 14030	0.265
1.270	0.90250 30645	-0.10258 31932	-0.20376 88437	1.17131 98301	0.270
1.275	0.90159 71994	-0.10358 74224	-0.19792 81118	1.16498 37821	0.275
1.280	0.90071 84765	-0.10456 25269	-0.19211 88983	1.15871 22990	0.280
1.285	0.89986 66769	-0.10550 86634	-0.18634 08828	1.15250 44385	0.285
1.290	0.89904 15863	-0.10642 59872	-0.18059 37494	1.14635 92764	0.290
1.295	0.89824 29947	-0.10731 46519	-0.17487 71870	1.14027 59053	0.295
1.300	0.89747 06963	-0.10817 48095	-0.16919 08889	1.13425 34350	0.300
1.305	0.89672 44895	-0.10900 66107	-0.16353 45526	1.12829 09915	0.305
1.310	0.89600 41767	-0.10981 02045	-0.15790 78803	1.12238 77175	0.310
1.315	0.89530 95644	-0.11058 57384	-0.15231 05782	1.11654 27706	0.315
1.320	0.89464 04630	-0.11133 33587	-0.14674 23568	1.11075 53246	0.320
1.325	0.89399 66866	-0.11205 32100	-0.14120 29305	1.10502 45678	0.325
1.330	0.89337 80535	-0.11274 54356	-0.13569 20180	1.09934 97037	0.330
1.335	0.89278 43850	-0.11341 01772	-0.13020 93416	1.09372 99497	0.335
1.340	0.89221 55072	-0.11404 75756	-0.12475 46279	1.08816 45379	0.340
1.345	0.89167 12485	-0.11465 77697	-0.11932 76069	1.08265 27136	0.345
1.350	0.89115 14420	-0.11524 08974	-0.11392 80127	1.07719 37361	0.350
1.355	0.89065 59235	-0.11579 70951	-0.10855 55827	1.07178 68773	0.355
1.360	0.89018 45324	-0.11632 64980	-0.10321 00582	1.06643 14226	0.360
1.365	0.88973 71116	-0.11682 92401	-0.09789 11840	1.06112 66696	0.365
1.370	0.88931 35074	-0.11730 54539	-0.09259 87082	1.05587 19286	0.370
1.375	0.88891 35692	-0.11775 52707	-0.08733 23825	1.05066 65216	0.375
1.380	0.88853 71494	-0.11817 88209	-0.08209 19619	1.04550 97829	0.380
1.385	0.88818 41041	-0.11857 62331	-0.07687 72046	1.04040 10578	0.385
1.390	0.88785 42918	-0.11894 76353	-0.07168 78723	1.03533 97036	0.390
1.395	0.88754 75748	-0.11929 31538	-0.06652 37297	1.03032 50881	0.395
1.400	0.88726 38175	-0.11961 29142	-0.06138 45446	1.02535 65905	0.400
1.405	0.88700 28884	-0.11990 70405	-0.05627 00879	1.02043 36002	0.405
1.410	0.88676 46576	-0.12017 56559	-0.05118 01337	1.01555 55173	0.410
1.415	0.88654 89993	-0.12041 88823	-0.04611 44589	1.01072 17518	0.415
1.420	0.88635 57896	-0.12063 68406	-0.04107 28433	1.00593 17241	0.420
1.425	0.88618 49081	-0.12082 96505	-0.03605 50697	1.00118 48640	0.425
1.430	0.88603 62361	-0.12099 74307	-0.03106 09237	0.99648 06113	0.430
1.435	0.88590 96587	-0.12114 02987	-0.02609 01935	0.99181 84147	0.435
1.440	0.88580 50635	-0.12125 83713	-0.02114 26703	0.98719 77326	0.440
1.445	0.88572 23397	-0.12135 17638	-0.01621 81479	0.98261 80318	0.445
1.450	0.88566 13803	-0.12142 05907	-0.01131 64226	0.97807 87886	0.450
1.455	0.88562 20800	-0.12146 49657	-0.00643 72934	0.97357 94874	0.455
1.460	0.88560 43364	-0.12148 50010	-0.00158 05620	0.96911 96215	0.460
1.465	0.88560 80495	-0.12148 08083	+0.00325 39677	0.96469 86921	0.465
1.470	0.88563 31217	-0.12145 24980	0.00806 64890	0.96031 62091	0.470
1.475	0.88567 94575	-0.12140 01797	0.01285 71930	0.95597 16896	0.475
1.480	0.88574 69646	-0.12132 39621	0.01762 62684	0.95166 46592	0.480
1.485	0.88583 55520	-0.12122 39528	0.02237 39013	0.94739 46509	0.485
1.490	0.88594 51316	-0.12110 02585	0.02710 02758	0.94316 12052	0.490
1.495	0.88607 56174	-0.12095 29852	0.03180 55736	0.93896 38700	0.495
1.500	0.88622 69255	-0.12078 22376	0.03648 99740	0.93480 22005	0.500
	$y!$	$\ln y!$	* $\frac{d}{dy}\ln y!$	* $\frac{d^2}{dy^2}\ln y!$	y
	$\left[{(-6)4 \atop 5}\right]$	$\left[{(-6)4 \atop 4}\right]$	$\left[{(-6)4 \atop 5}\right]$	$\left[{(-6)9 \atop 5}\right]$	

$\log_{10} e = 0.43429\ 44819$

*See page II.

Table I (Cont'd)

GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS Table 6.1

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.500	0.88622 69255	-0.12078 22376	0.03648 99740	0.93480 22005	0.500
1.505	0.88639 89744	-0.12058 81200	0.04115 36543	0.93067 57588	0.505
1.510	0.88659 16850	-0.12037 07353	0.04579 67896	0.92658 41142	0.510
1.515	0.88680 49797	-0.12013 01860	0.05041 95527	0.92252 68425	0.515
1.520	0.88703 87833	-0.11986 65735	0.05502 21146	0.91850 35265	0.520
1.525	0.88729 30231	-0.11957 99983	0.05960 46439	0.91451 37552	0.525
1.530	0.88756 76278	-0.11927 05601	0.06416 73074	0.91055 71245	0.530
1.535	0.88786 25287	-0.11893 83580	0.06871 02697	0.90663 32361	0.535
1.540	0.88817 76586	-0.11858 34900	0.07323 36936	0.90274 16984	0.540
1.545	0.88851 29527	-0.11820 60534	0.07773 77400	0.89888 21253	0.545
1.550	0.88886 83478	-0.11780 61446	0.08222 25675	0.89505 41371	0.550
1.555	0.88924 37830	-0.11738 38595	0.08668 83334	0.89125 73596	0.555
1.560	0.88963 91990	-0.11693 92928	0.09113 51925	0.88749 14249	0.560
1.565	0.89005 45387	-0.11647 25388	0.09556 32984	0.88375 59699	0.565
1.570	0.89048 97463	-0.11598 36908	0.09997 28024	0.88005 06378	0.570
1.575	0.89094 47686	-0.11547 28415	0.10436 38544	0.87637 50766	0.575
1.580	0.89141 95537	-0.11494 00828	0.10873 66023	0.87272 89402	0.580
1.585	0.89191 40515	-0.11438 55058	0.11309 11923	0.86911 18871	0.585
1.590	0.89242 82141	-0.11380 92009	0.11742 77690	0.86552 35815	0.590
1.595	0.89296 19949	-0.11321 12579	0.12174 64754	0.86196 36921	0.595
1.600	0.89351 53493	-0.11259 17657	0.12604 74528	0.85843 18931	0.600
1.605	0.89408 82342	-0.11195 08127	0.13033 08407	0.85492 78630	0.605
1.610	0.89468 06085	-0.11128 84864	0.13459 67772	0.85145 12856	0.610
1.615	0.89529 24327	-0.11060 48737	0.13884 53988	0.84800 18488	0.615
1.620	0.89592 36685	-0.10990 00610	0.14307 68404	0.84457 92455	0.620
1.625	0.89657 42800	-0.10917 41338	0.14729 12354	0.84118 31730	0.625
1.630	0.89724 42326	-0.10842 71769	0.15148 87158	0.83781 33330	0.630
1.635	0.89793 34930	-0.10765 92746	0.15566 94120	0.83446 94315	0.635
1.640	0.89864 20302	-0.10687 05105	0.15983 34529	0.83115 11790	0.640
1.645	0.89936 98138	-0.10606 09676	0.16398 09660	0.82785 82897	0.645
1.650	0.90011 68163	-0.10523 07282	0.16811 20776	0.82459 04826	0.650
1.655	0.90088 30104	-0.10437 98739	0.17222 69122	0.82134 74802	0.655
1.660	0.90166 83712	-0.10350 84860	0.17632 55933	0.81812 90092	0.660
1.665	0.90247 28748	-0.10261 66447	0.18040 82427	0.81493 48001	0.665
1.670	0.90329 64995	-0.10170 44301	0.18447 49813	0.81176 45875	0.670
1.675	0.90413 92243	-0.10077 19212	0.18852 59282	0.80861 81094	0.675
1.680	0.90500 10302	-0.09981 91969	0.19256 12015	0.80549 51079	0.680
1.685	0.90588 18996	-0.09884 63351	0.19658 09180	0.80239 53282	0.685
1.690	0.90678 18160	-0.09785 34135	0.20058 51931	0.79931 85198	0.690
1.695	0.90770 07650	-0.09684 05088	0.20457 41410	0.79626 44350	0.695
1.700	0.90863 87329	-0.09580 76974	0.20854 78749	0.79323 28302	0.700
1.705	0.90959 57079	-0.09475 50552	0.21250 65064	0.79022 34645	0.705
1.710	0.91057 16796	-0.09368 26573	0.21645 01462	0.78723 61012	0.710
1.715	0.91156 66390	-0.09259 05785	0.22037 89037	0.78427 05060	0.715
1.720	0.91258 05779	-0.09147 88929	0.22429 28871	0.78132 64486	0.720
1.725	0.91361 34904	-0.09034 76741	0.22819 22037	0.77840 37011	0.725
1.730	0.91466 53712	-0.08919 69951	0.23207 69593	0.77550 20396	0.730
1.735	0.91573 62171	-0.08802 69286	0.23594 72589	0.77262 12424	0.735
1.740	0.91682 60252	-0.08683 75466	0.23980 32061	0.76976 10915	0.740
1.745	0.91793 47950	-0.08562 89203	0.24364 49038	0.76692 13714	0.745
1.750	0.91906 25268	-0.08440 11210	0.24747 24535	0.76410 18699	0.750
	$y!$	$\ln y!$	$\frac{d}{dy}\ln y!$	$\frac{d^2}{dy^2}\ln y!$	y
	$\left[{(-6)3 \atop 4}\right]$	$\left[{(-6)3 \atop 4}\right]$	$\left[{(-6)3 \atop 4}\right]$	$\left[{(-6)4 \atop 5}\right]$	

$\log_{10} e = 0.43429\ 44819$

Table 6.1 GAMMA, DIGAMMA AND TRIGAMMA FUNCTIONS

x	$\Gamma(x)$	$\ln \Gamma(x)$	$\psi(x)$	$\psi'(x)$	
1.750	0.91906 25268	-0.08440 11210	0.24747 24535	0.76410 18699	0.750
1.755	0.92020 92224	-0.08315 42192	0.25128 59559	0.76130 23773	0.755
1.760	0.92137 48846	-0.08188 82847	0.25508 55103	0.75852 26870	0.760
1.765	0.92255 95178	-0.08060 33871	0.25887 12154	0.75576 25950	0.765
1.770	0.92376 31277	-0.07929 95955	0.26264 31686	0.75302 19003	0.770
1.775	0.92498 57211	-0.07797 69782	0.26640 14664	0.75030 04040	0.775
1.780	0.92622 73062	-0.07663 56034	0.27014 62043	0.74759 79107	0.780
1.785	0.92748 78926	-0.07527 55386	0.27387 74769	0.74491 42268	0.785
1.790	0.92876 74904	-0.07389 68509	0.27759 53776	0.74224 91617	0.790
1.795	0.93006 61123	-0.07249 96070	0.28129 99992	0.73960 25271	0.795
1.800	0.93138 37710	-0.07108 38729	0.28499 14333	0.73697 41375	0.800
1.805	0.93272 04811	-0.06964 97145	0.28866 97707	0.73436 38093	0.805
1.810	0.93407 62585	-0.06819 71969	0.29233 51012	0.73177 13620	0.810
1.815	0.93545 11198	-0.06672 63850	0.29598 75138	0.72919 66166	0.815
1.820	0.93684 50832	-0.06523 73431	0.29962 70966	0.72663 93972	0.820
1.825	0.93825 81682	-0.06373 01353	0.30325 39367	0.72409 95297	0.825
1.830	0.93969 03951	-0.06220 48248	0.30686 81205	0.72157 68426	0.830
1.835	0.94114 17859	-0.06066 14750	0.31046 97335	0.71907 11662	0.835
1.840	0.94261 23634	-0.05910 01483	0.31405 88602	0.71658 23333	0.840
1.845	0.94410 21519	-0.05752 09071	0.31763 55846	0.71411 01788	0.845
1.850	0.94561 11764	-0.05592 38130	0.32119 99895	0.71165 45396	0.850
1.855	0.94713 94637	-0.05430 89276	0.32475 21572	0.70921 52546	0.855
1.860	0.94868 70417	-0.05267 63117	0.32829 21691	0.70679 21650	0.860
1.865	0.95025 39389	-0.05102 60260	0.33182 01056	0.70438 51138	0.865
1.870	0.95184 01855	-0.04935 81307	0.33533 60467	0.70199 39461	0.870
1.875	0.95344 58127	-0.04767 26854	0.33884 00713	0.69961 85089	0.875
1.880	0.95507 08530	-0.04596 97497	0.34233 22577	0.69725 86512	0.880
1.885	0.95671 53398	-0.04424 93824	0.34581 26835	0.69491 42236	0.885
1.890	0.95837 93077	-0.04251 16423	0.34928 14255	0.69258 50790	0.890
1.895	0.96006 27927	-0.04075 65875	0.35273 85596	0.69027 10717	0.895
1.900	0.96176 58319	-0.03898 42759	0.35618 41612	0.68797 20582	0.900
1.905	0.96348 84632	-0.03719 47650	0.35961 83049	0.68568 78965	0.905
1.910	0.96523 07261	-0.03538 81118	0.36304 10646	0.68341 84465	0.910
1.915	0.96699 26608	-0.03356 43732	0.36645 25136	0.68116 35696	0.915
1.920	0.96877 43090	-0.03172 36054	0.36985 27244	0.67892 31293	0.920
1.925	0.97057 57134	-0.02986 58646	0.37324 17688	0.67669 69903	0.925
1.930	0.97239 69178	-0.02799 12062	0.37661 97179	0.67448 50194	0.930
1.935	0.97423 79672	-0.02609 96858	0.37998 66424	0.67228 70846	0.935
1.940	0.97609 89075	-0.02419 13581	0.38334 26119	0.67010 30559	0.940
1.945	0.97797 97861	-0.02226 62778	0.38668 76959	0.66793 28044	0.945
1.950	0.97988 06513	-0.02032 44991	0.39002 19627	0.66577 62034	0.950
1.955	0.98180 15524	-0.01836 60761	0.39334 54805	0.66363 31270	0.955
1.960	0.98374 25404	-0.01639 10621	0.39665 83163	0.66150 34514	0.960
1.965	0.98570 36664	-0.01439 95106	0.39996 05371	0.65938 70538	0.965
1.970	0.98768 49838	-0.01239 14744	0.40325 22088	0.65728 38134	0.970
1.975	0.98968 65462	-0.01036 70060	0.40653 33970	0.65519 36104	0.975
1.980	0.99170 84087	-0.00832 61578	0.40980 41664	0.65311 63266	0.980
1.985	0.99375 06274	-0.00626 89816	0.41306 45816	0.65105 18450	0.985
1.990	0.99581 32598	-0.00419 55291	0.41631 47060	0.64900 00505	0.990
1.995	0.99789 63643	-0.00210 58516	0.41955 46030	0.64696 08286	0.995
2.000	1.00000 00000	0.00000 00000	0.42278 43351	0.64493 40668	1.000
	$y!$	$\ln y!$	$\frac{d}{dy} \ln y!$	$\frac{d^2}{dy^2} \ln y!$	y
	$\left[\begin{matrix}(-6)2\\4\end{matrix}\right]$	$\left[\begin{matrix}(-6)2\\4\end{matrix}\right]$	$\left[\begin{matrix}(-6)2\\4\end{matrix}\right]$	$\left[\begin{matrix}(\ 6)2\\4\end{matrix}\right]$	

$\log_{10} e = 0.43429\ 44819$

Table II GAMMA FUNCTION (α: INTEGER)

Table 6.3 GAMMA AND DIGAMMA FUNCTIONS FOR INTEGER AND HALF-INTEGER VALUES

n	$\Gamma(n)$	$1/\Gamma(n)$	$\Gamma(n+\frac{1}{2})$	$\psi(n)$	$f_1(n)$	$f_3(n)$
1	(0)1.00000 00000	(0)1.00000 000	(-1)8.86226 93	-0.57721 56649	1.08443 755	0.57721 566
2	(0)1.00000 00000	(0)1.00000 000	(0)1.32934 04	+0.42278 43351	1.04220 712	0.27036 285
3	(0)2.00000 00000	(- 1)5.00000 000	(0)3.32335 10	0.92278 43351	1.02806 452	0.17582 795
4	(0)6.00000 00000	(- 1)1.66666 667	(1)1.16317 28	1.25611 76684	1.02100 830	0.13017 669
5	(1)2.40000 00000	(- 2)4.16666 667	(1)5.23427 78	1.50611 76684	1.01678 399	0.10332 024
6	(2)1.20000 00000	(- 3)8.33333 333	(2)2.87885 28	1.70611 76684	1.01397 285	0.08564 180
7	(2)7.20000 00000	(- 3)1.38888 889	(3)1.87125 43	1.87278 43351	1.01196 776	0.07312 581
8	(3)5.04000 00000	(- 4)1.98412 698	(4)1.40344 07	2.01564 14780	1.01046 565	0.06380 006
9	(4)4.03200 00000	(- 5)2.48015 873	(5)1.19292 46	2.14064 14780	1.00929 843	0.05658 310
10	(5)3.62880 00000	(- 6)2.75573 192	(6)1.13327 84	2.25175 25891	1.00836 536	0.05083 250
11	(6)3.62880 00000	(- 7)2.75573 192	(7)1.18994 23	2.35175 25891	1.00760 243	0.04614 268
12	(7)3.99168 00000	(- 8)2.50521 084	(8)1.36843 37	2.44266 16800	1.00696 700	0.04224 497
13	(8)4.79001 60000	(- 9)2.08767 570	(9)1.71054 21	2.52599 50133	1.00642 958	0.03895 434
14	(9)6.22702 08000	(-10)1.60590 438	(10)2.30923 18	2.60291 80902	1.00596 911	0.03613 924
15	(10)8.71782 91200	(-11)1.14707 456	(11)3.34838 61	2.67434 66617	1.00557 019	0.03370 354
16	(12)1.30767 43680	(-13)7.64716 373	(12)5.18999 85	2.74101 33283	1.00522 124	0.03157 539
17	(13)2.09227 89888	(-14)4.77947 733	(13)8.56349 74	2.80351 33283	1.00491 343	0.02970 002
18	(14)3.55687 42810	(-15)2.81145 725	(15)1.49861 21	2.86233 68577	1.00463 988	0.02803 490
19	(15)6.40237 37057	(-16)1.56192 070	(16)2.77243 23	2.91789 24133	1.00439 519	0.02654 657
20	(17)1.21645 10041	(-18)8.22063 525	(17)5.40624 30	2.97052 39922	1.00417 501	0.02520 828
21	(18)2.43290 20082	(-19)4.11031 762	(19)1.10827 98	3.02052 39922	1.00397 584	0.02399 845
22	(19)5.10909 42172	(-20)1.95729 411	(20)2.38280 16	3.06814 30399	1.00379 480	0.02289 941
23	(21)1.12400 07278	(-22)8.89679 139	(21)5.36130 3[illegible]	3.11359 75853	1.00362 953	0.02189 663
24	(22)2.58520 16739	(-23)3.86817 017	(23)1.25990 63	3.15707 58462	1.00347 806	0.02097 798
25	(23)6.20448 40173	(-24)1.61173 757	(24)3.08677 05	3.19874 25129	1.00333 872	0.02013 331
26	(25)1.55112 10043	(-26)6.44695 029	(25)7.87126 49	3.23874 25129	1.00321 011	0.01935 403
27	(26)4.03291 46113	(-27)2.47959 626	(27)2.08588 52	3.27720 40513	1.00309 105	0.01863 281
28	(28)1.08888 69450	(-29)9.18368 986	(28)5.73618 43	3.31424 10884	1.00298 050	0.01796 342
29	(29)3.04888 34461	(-30)3.27988 924	(30)1.63481 25	3.34995 53741	1.00287 758	0.01734 046
30	(30)8.84176 19937	(-31)1.13099 629	(31)4.82269 69	3.38443 81327	1.00278 154	0.01675 925
31	(32)2.65252 85981	(-33)3.76998 763	(33)1.47092 26	3.41777 14660	1.00269 170	0.01621 574
32	(33)8.22283 86542	(-34)1.21612 504	(34)4.63340 61	3.45002 95305	1.00260 748	0.01570 637
33	(35)2.63130 83693	(-36)3.80039 076	(36)1.50585 70	3.48127 95305	1.00252 837	0.01522 803
34	(36)8.68331 76188	(-37)1.15163 356	(37)5.04462 09	3.51158 25608	1.00245 392	0.01477 796
35	(38)2.95232 79904	(-39)3.38715 754	(39)1.74039 42	3.54099 43255	1.00238 372	0.01435 374
36	(40)1.03331 47966	(-41)9.67759 296	(40)6.17839 94	3.56956 57541	1.00231 744	0.01395 318
37	(41)3.71993 32679	(-42)2.68822 027	(42)2.25511 58	3.59734 35319	1.00225 474	0.01357 438
38	(43)1.37637 53091	(-44)7.26546 018	(43)8.45668 42	3.62437 05589	1.00219 534	0.01321 560
39	(44)5.23022 61747	(-45)1.91196 320	(45)3.25582 34	3.65068 63484	1.00213 899	0.01287 530
40	(46)2.03978 82081	(-47)4.90246 976	(47)1.28605 02	3.67632 73740	1.00208 546	0.01255 208
41	(47)8.15915 28325	(-48)1.22561 744	(48)5.20850 35	3.70132 73740	1.00203 455	0.01224 469
42	(49)3.34525 26613	(-50)2.98931 083	(50)2.16152 90	3.72571 76179	1.00198 606	0.01195 200
43	(51)1.40500 61178	(-52)7.11740 673	(51)9.18649 81	3.74952 71417	1.00193 983	0.01167 297
44	(52)6.04152 63063	(-53)1.65521 087	(53)3.99612 67	3.77278 29557	1.00189 570	0.01140 668
45	(54)2.65827 15748	(-55)3.76184 288	(55)1.77827 64	3.79551 02284	1.00185 354	0.01115 226
46	(56)1.19622 22087	(-57)8.35965 084	(56)8.09115 74	3.81773 24506	1.00181 321	0.01090 895
47	(57)5.50262 21598	(-58)1.81731 540	(58)3.76238 82	3.83947 15811	1.00177 460	0.01067 602
48	(59)2.58623 24151	(-60)3.86662 851	(60)1.78713 44	3.86074 81768	1.00173 759	0.01045 283
49	(61)1.24139 15593	(-62)8.05547 607	(61)8.66760 18	3.88158 15102	1.00170 210	0.01023 879
50	(62)6.08281 86403	(-63)1.64397 471	(63)4.29046 29	3.90198 96734	1.00166 803	0.01003 333
51	(64)3.04140 93202	(-65)3.28794 942	(65)2.16668 38	3.92198 96734	1.00163 530	0.00983 596
	$(n-1)!$	$1/(n-1)!$	$(n-\frac{1}{2})!$	$\frac{d}{dn}\ln(n-1)!$ *		

$n! = (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n} f_1(n)$ $\Gamma(n) = (2\pi)^{\frac{1}{2}} n^{n-\frac{1}{2}} e^{-n} f_1(n)$ $\psi(n) = \ln n - f_3(n)$ $(2\pi)^{\frac{1}{2}} = 2.50662\ 82746\ 31001$

$\psi(n)$ compiled from H. T. Davis, Tables of the higher mathematical functions, 2 vols. (Principia Press, Bloomington, Ind., 1933, 1935) (with permission).

*See page 11.

TABLE II (Cont'd)

GAMMA AND DIGAMMA FUNCTIONS FOR INTEGER AND HALF-INTEGER VALUES Table 6.3

n	$\Gamma(n)$	$1/\Gamma(n)$	$\Gamma(n+\frac{1}{2})$	$\psi(n)$	$f_1(n)$	$f_3(n)$
51	(64)3.04140 93202	(- 65)3.28794 942	(65)2.16668 38	3.92198 96734	1.00163 530	0.00983 596
52	(66)1.55111 87533	(- 67)6.44695 964	(67)1.11584 21	3.94159 75166	1.00160 383	0.00964 620
53	(67)8.06581 75171	(- 68)1.23979 993	(68)5.85817 12	3.96082 82858	1.00157 355	0.00946 363
54	(69)4.27488 32841	(- 70)2.33924 515	(70)3.13412 16	3.97969 62103	1.00154 438	0.00928 784
55	(71)2.30843 69734	(- 72)4.33193 547	(72)1.70809 63	3.99821 47288	1.00151 628	0.00911 846
56	(73)1.26964 03354	(- 74)7.87624 631	(73)9.47993 44	4.01639 65470	1.00148 919	0.00895 514
57	(74)7.10998 58780	(- 75)1.40647 255	(75)5.35616 29	4.03425 36899	1.00146 304	0.00879 758
58	(76)4.05269 19505	(- 77)2.46749 571	(77)3.07979 37	4.05179 75495	1.00143 780	0.00864 546
59	(78)2.35056 13313	(- 79)4.25430 295	(79)1.80167 93	4.06903 89288	1.00141 341	0.00849 852
60	(80)1.38683 11855	(- 81)7.21068 296	(81)1.07199 92	4.08598 80814	1.00138 984	0.00835 648
61	(81)8.32098 71127	(- 82)1.20178 049	(82)6.48559 51	4.10265 47481	1.00136 704	0.00821 912
62	(83)5.07580 21388	(- 84)1.97013 196	(84)3.98864 10	4.11904 81907	1.00134 498	0.00808 619
63	(85)3.14699 73260	(- 86)3.17763 219	(86)2.49290 06	4.13517 72229	1.00132 362	0.00795 750
64	(87)1.98260 83154	(- 88)5.04386 062	(88)1.58299 19	4.15105 02388	1.00130 292	0.00783 284
65	(89)1.26886 93219	(- 90)7.88103 221	(90)1.02102 98	4.16667 52388	1.00128 286	0.00771 203
66	(90)8.24765 05921	(- 91)1.21246 649	(91)6.68774 50	4.18205 98542	1.00126 341	0.00759 489
67	(92)5.44344 93908	(- 93)1.83707 044	(93)4.44735 04	4.19721 13693	1.00124 455	0.00748 125
68	(94)3.64711 10918	(- 95)2.74189 619	(95)3.00196 15	4.21213 67425	1.00122 623	0.00737 096
69	(96)2.48003 55424	(- 97)4.03220 028	(97)2.05634 36	4.22684 26248	1.00120 845	0.00726 388
70	(98)1.71122 45243	(- 99)5.84376 852	(99)1.42915 88	4.24133 53785	1.00119 118	0.00715 986
71	(100)1.19785 71670	(-101)8.34824 074	(101)1.00755 70	4.25562 10927	1.00117 439	0.00705 878
72	(101)8.50478 58857	(-102)1.17580 856	(102)7.20403 24	4.26970 55998	1.00115 807	0.00696 052
73	(103)6.12344 58377	(-104)1.63306 744	(104)5.22292 35	4.28359 44887	1.00114 220	0.00686 495
74	(105)4.47011 54615	(-106)2.23707 868	(106)3.83884 87	4.29729 31188	1.00112 675	0.00677 197
75	(107)3.30788 54415	(-108)3.02307 930	(108)2.85994 23	4.31080 66323	1.00111 172	0.00668 148
76	(109)2.48091 40811	(-110)4.03077 240	(110)2.15925 64	4.32413 99657	1.00109 709	0.00659 337
77	(111)1.88549 47017	(-112)5.30364 789	(112)1.65183 12	4.33729 78604	1.00108 283	0.00650 756
78	(113)1.45183 09203	(-114)6.88785 441	(114)1.28016 92	4.35028 48734	1.00106 894	0.00642 395
79	(115)1.13242 81178	(-116)8.83058 257	(116)1.00493 28	4.36310 53862	1.00105 540	0.00634 247
80	(116)8.94618 21308	(-117)1.11779 526	(117)7.98921 57	4.37576 36140	1.00104 220	0.00626 302
81	(118)7.15694 57046	(-119)1.39724 408	(119)6.43131 87	4.38826 36140	1.00102 933	0.00618 554
82	(120)5.79712 60207	(-121)1.72499 269	(121)5.24152 47	4.40060 92931	1.00101 677	0.00610 995
83	(122)4.75364 33370	(-123)2.10364 962	(123)4.32425 79	4.41280 44150	1.00100 452	0.00603 619
84	(124)3.94552 39697	(-125)2.53451 761	(125)3.61075 53	4.42485 26078	1.00099 255	0.00596 419
85	(126)3.31424 01346	(-127)3.01728 287	(127)3.05108 83	4.43675 73697	1.00098 087	0.00589 389
86	(128)2.81710 41144	(-129)3.54974 456	(129)2.60868 05	4.44852 20756	1.00096 946	0.00582 522
37	(130)2.42270 95384	(-131)4.12760 995	(131)2.25650 86	4.46014 99825	1.00095 831	0.00575 814
88	(132)2.10775 72984	(-133)4.74437 926	(133)1.97444 50	4.47164 42354	1.00094 741	0.00569 258
89	(134)1.85482 64226	(-135)5.39134 006	(135)1.74738 38	4.48300 78718	1.00093 676	0.00562 850
90	(136)1.65079 55161	(-137)6.05768 546	(137)1.56390 85	4.49424 38268	1.00092 635	0.00556 584
91	(138)1.48571 59645	(-139)6.73076 163	(139)1.41533 72	4.50535 49379	1.00091 617	0.00550 457
92	(140)1.35200 15277	(-141)7.39644 134	(141)1.29503 36	4.51634 39489	1.00090 620	0.00544 463
93	(142)1.24384 14055	(-143)8.03961 016	(143)1.19790 60	4.52721 35142	1.00089 646	0.00538 598
94	(144)1.15677 25071	(-145)8.64474 211	(145)1.12004 22	4.53796 62023	1.00088 691	0.00532 858
95	(146)1.08736 61567	(-147)9.19653 415	(147)1.05843 98	4.54860 45002	1.00087 757	0.00527 239
96	(148)1.03299 78488	(-149)9.68056 227	(149)1.01081 00	4.55913 08160	1.00086 843	0.00521 738
97	(149)9.91677 93487	(-150)1.00839 190	(150)9.75431 69	4.56954 74827	1.00085 947	0.00516 350
98	(151)9.61927 59682	(-152)1.03957 928	(152)9.51045 90	4.57985 67610	1.00085 070	0.00511 072
99	(153)9.42689 04489	(-154)1.06079 519	(154)9.36780 21	4.59006 08426	1.00084 210	0.00505 901
100	(155)9.33262 15444	(-156)1.07151 029	(156)9.32096 31	4.60016 18527	1.00083 368	0.00500 833
101	(157)9.33262 15444	(-158)1.07151 029	(158)9.36756 79	4.61016 18527	1.00082 542	0.00495 866
	$(n-1)!$	$1/(n-1)!$	$(n-\frac{1}{2})!$	$*\frac{d}{dn}\ln(n-1)!$	$\begin{bmatrix}(-7)2\\3\end{bmatrix}$	$\begin{bmatrix}(-6)1\\4\end{bmatrix}$

$n! \sim (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n} f_1(n)$ $\Gamma(n) \sim (2\pi)^{\frac{1}{2}} n^{n-\frac{1}{2}} e^{-n} f_1(n)$ $\psi(n) \sim \ln n - f_3(n)$ $(2\pi)^{\frac{1}{2}} = 2.50662\ 82746\ 31001$

*See page II.

CALCULATOR METHOD

The expression $\Gamma(\alpha + 1) = \alpha\,\Gamma(\alpha)$ is applied repeatedly to increase the value of the argument until is it greater than 9, when Stirling's formula can be applied. If the argument is initially greater than 9, Stirling's formula is used at once.

Stirling's formula is given in Abramowitz [1]

For $\alpha > 9$:

Define
$$Y = \alpha + 1$$
$$X = (\alpha + 1)\alpha$$

$$S = 0.9189385332 - Y + (Y + 0.5)\ln Y + \frac{1}{(12Y)}\left[1 - \frac{1}{30Y^2} + \frac{1}{105Y^4}\right]$$

$$\Gamma(\alpha) = \frac{e^S}{X}$$

EXAMPLE

FIND: $\Gamma(10.4)$

SOLUTION: $Y = 11.4$; $X = (11.4)(10.4)$

$$S = 0.9189385332 - 11.4 + (11.9)\ \ln 11.4$$

$$+ \frac{1}{(12)(11.4)}\left[1 - \frac{1}{30(11.4)^2} + \frac{1}{105(11.4)^4}\right]$$

$S = 18.48624553$

$e^S = 1.067761911 \times 10^8$

$X = 118.56$

$$\Gamma(10.4) = \frac{e^S}{X} = 900608.9$$

FOR $\alpha < 9$:

DEFINE $Y = 9 +$ fractional part of α

$$X = \alpha(\alpha+1)(\alpha+2) \times \ldots Y$$

$$S = 0.9189385332 - Y + (Y + 0.5)\ln Y$$

$$+ \frac{1}{(12Y)}\left[1 - \frac{1}{30Y^2} + \frac{1}{150Y^4}\right]$$

$$\Gamma(\alpha) = \frac{e^S}{X}$$

EXAMPLE

FIND: Γ (2.4)

SOLUTION:

$Y = 9.4$

$X = 9.4 \times 8.4 \times 7.4 \times 6.4 \times 5.4 \times 4.4 \times 3.4 \times 2.4$

$X = 725029.0842$

$$S = 0.9189385332 - 9.4 + 9.9 \ln 9.4 + \frac{1}{(12)(9.4)}\left[1 - \frac{1}{30(9.4)^2} + \frac{1}{150(9.4)^4}\right]$$

$S = 13.71082637$

$$\Gamma(2.4) = \frac{e^S}{X} = \frac{9.006089021 \times 10^5}{7.250290842 \times 10^5}$$

$\Gamma(2.4) = 1.242169346$

AS A CHECK

$\Gamma(2.4) = 1.4\ \Gamma(1.4) = 1.4 \times 0.8872638175$

See TABLE I

$\Gamma(2.4) = 1.242169345$

BASIC LANGUAGE PROGRAM

The previous calculator method is shown as a program listing in Basic Language, PL-1.

PL-1

PROGRAM LISTING FOR $\Gamma(\alpha)$ IN BASIC LANGUAGE

```
10 REM A= ALPHA
20 REM G= GAMMA FUNCTION WITH ARGUMENT ALPHA
30 A=5.44
40 Y=A
50 X=A
60 Y=Y+1
70 X=X*Y
80 D=Y-9
90 IF D>=0 THEN 110
100 GO TO 60
110 S=.9189385332
120 S=S+(Y-.5)*LOG(Y)-Y
130 V=1-(1/(30*Y↑2))+(1/(105*Y↑4))
140 V=(1/(12*Y))*V
150 S=S+V
160 G=EXP(S)/X
170 PRINT A,G
180 END
```

TI-59 Methods

A. USER ENTERED PROGRAM

The following program PL-2 is for use with the TI-59 Programmable Calculator and is part of Texas Instrument's Math 39 program exchange. It calculates the Gamma Function $\Gamma(x)$ for integer and non-integer values of the agrument x.

An internal accuracy check using a fractional argument can be made by calculating $\Gamma(0.5)$, then squaring the answer. The result should be π. Subtracting the stored value of π from the previously calculated value gives 4×10^{-10}. Thus, the error is 4 in the 11th digit. Other non-integer values have been compared with the National Bureau of Standards Tables [1] and almost all agree within +/- one unit in the 10th digit. PL-2 accuracy is considered accurate for all practical purposes.

To use enter the value of x; then press [D]. The computed value of $\Gamma(x)$ will ultimately be displayed.

$x < 69.5$

Calculation time is 5 to 15 seconds, depending on x.

EXAMPLE: Compute $\Gamma(6.5)$

Enter 6.5. PRESS [D].

The displayed output is 287.8852778

Thus, $\Gamma(6.5) = 287.8852778$

TI-59 Methods (Cont'd)

B. MATH UTILITY MODULE PROGRAM MU-11

The new math utility module can be used for computing $\Gamma(x)$ directly.

EXAMPLE: Compute $\Gamma(6.3)$

ENTER	PRESS	DISPLAY
	[2nd] [Pgm] 11	
6.3	[A]	201.8132752

Thus, $\Gamma(6.3) = 201.8132752$

PL-2

GAMMA FUNCTION;

LISTING FOR TI-59 PROGRAMMABLE CALCULATOR

LOC	CODE	KEY
000	76	LBL
001	14	D
002	42	STO
003	00	00
004	42	STO
005	01	01
006	68	NOP
007	69	OP
008	21	21
009	43	RCL
010	00	00
011	65	×
012	43	RCL
013	01	01
014	95	=
015	42	STO
016	00	00
017	43	RCL
018	01	01
019	75	-
020	09	9
021	95	=
022	77	GE
023	00	00
024	29	29
025	61	GTO
026	00	00
027	07	07
028	68	NOP
029	53	(
030	53	(
031	02	2
032	65	×
033	89	π
034	54	)
035	23	LNX
036	65	×
037	93	.
038	05	5
039	85	+
040	53	(
041	43	RCL
042	01	01
043	85	+
044	93	.
045	05	5
046	54	)
047	65	×
048	43	RCL
049	01	01
050	23	LNX
051	75	-
052	43	RCL
053	01	01
054	85	+
055	43	RCL
056	01	01
057	35	1/X
058	55	÷
059	01	1
060	02	2
061	65	×
062	53	(
063	01	1
064	75	-
065	53	(
066	43	RCL
067	01	01
068	33	X²
069	65	×
070	03	3
071	00	0
072	54	)
073	35	1/X
074	85	+
075	53	(
076	43	RCL
077	01	01
078	33	X²
079	33	X²
080	65	×
081	01	1
082	00	0
083	05	5
084	54	)
085	35	1/X
086	54	)
087	54	)
088	22	INV
089	23	LNX
090	55	÷
091	43	RCL
092	00	00
093	95	=
094	91	R/S

HP-67 Method

A program listing is shown (PL-3) for computing $\Gamma(\alpha)$ on an HP-67 Programmable Calculator using the previously described Stirling's approximation formula.

GAMMA FUNCTION

→ Γ(A) | R/S →A

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
1	ENTER ARGUMENT VALUE	A			A
2	PRESS A			A	Γ(A)
3	PRESS R/S				A
	FOR NEW CASE GO TO STEP 1				
	(OVER)				

(OVER)

PL-3 (CONT'D)

STEP	KEY ENTRY	KEY CODE	COMMENTS
	LBLA	31 25 11	
	STO1	33 01	
	STO2	33 02	
	STO3	33 03	
	LBLB	31 25 12	
	1	01	
	STO+2	33 61 02	
	RCL2	34 02	
	RCL3	34 03	
010	x	71	
	STO3	33 03	
	RCL2	34 02	
	9	09	
	−	51	
	x < 0	31 71	
	GTO B	22 12	
	.	83	
	9	09	
	1	01	
020	8	08	
	9	09	
	3	03	
	8	08	
	5	05	
	3	03	
	3	03	
	2	02	
	RCL2	34 02	
	−	51	
030	RCL2	34 02	
	f LN	31 52	
	RCL2	34 02	
	.	83	
	5	05	
	+	61	
	x	71	
	+	61	
	STO 4	33 04	
	RCL2	34 02	
040	g x^2	32 54	
	3	03	
	0	00	
	x	71	
	h 1/x	35 62	
	CHS	42	
	1	01	
	+	61	
	RCL2	34 02	
	4	04	
050	h y^x	35 63	
	1	01	
	0	00	
	5	05	
	x	71	
	h 1/x	35 62	
	+	61	
	STO5	33 05	
	RCL 2	34 02	
	1	01	
060	2	02	
	x	71	
	h 1/x	35 62	
	RCL5	34 05	
	x	71	
	RCL4	34 04	
	+	61	
	g e^x	32 52	
	RCL3	34 03	
	÷	81	
070	R/S	84	
	RCL1	34 01	
	RTN	35 22	
080			
090			
100			
110			

REGISTERS

0	1	2	3	4	5	6	7	8	9
	A	Y	X	S	V				
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I

HP-34C METHOD

The key x! can be used directly to calculate $\Gamma(x)$ using the following relationship:

$$\Gamma(x) = (x - 1)!$$

EXAMPLE: Calculate $\Gamma(6.3)$

KEYSTROKES	DISPLAY	COMMENT
6.3	6.3	x
ENTER ↑ 1[-]	5.3	x-1
[h] [x!]	201.8132	$\Gamma(6.3)$

B. INCOMPLETE GAMMA FUNCTIONS COMPUTATION

Definition:

The two Incomplete Gamma Functions $\gamma(\alpha,\tau)$ and $Q(\alpha,\tau)$ are defined as follows:

$$\gamma(\alpha,\tau) = \int_0^{\tau} x^{\alpha-1} e^{-x} dx$$

$$Q(\alpha,\tau) = \int_{\tau}^{\infty} x^{\alpha-1} e^{-x} dx$$

The Incomplete Gamma Functions are related to the Complete Gamma Function $\Gamma(\alpha)$ as follows:

$$\Gamma(\alpha) = \gamma(\alpha,\tau) + Q(\alpha,\tau)$$

That is

$$\int_0^{\infty} y \, dx = \int_0^{\tau} y \, dx + \int_{\tau}^{\infty} y \, dx$$

where $y = x^{\alpha-1} e^{-x}$

TABULAR METHOD

On page 941 of Abramowitz [1]

$$\gamma(\alpha,\tau) = \Gamma(\alpha)\ P\ (\chi^2|\nu)$$

where $\chi^2 = 2\tau$; $\nu = 2\alpha$

$$Q\ (\alpha,\tau) = \Gamma(\alpha) - \gamma(\alpha,\tau)$$

Also $Q\ (\alpha,\tau) = \Gamma(\alpha)\ Q\ (\chi^2|\ \nu)$

NOTE: $Q\ (\alpha,\tau) \neq Q\ (\chi^2|\nu)$

$$P\ (\chi^2|\nu) = 1 - Q\ (\chi^2|\nu)$$

Values of $Q\ (\chi^2|\nu)$ are tabulated in Table III, on pages 978-983 [1]

where $y = x^{\alpha-1}\ e^{-x}$

A pictorial representation is shown in figure 2. Note that y_{max} occurs at $x = \alpha-1$.

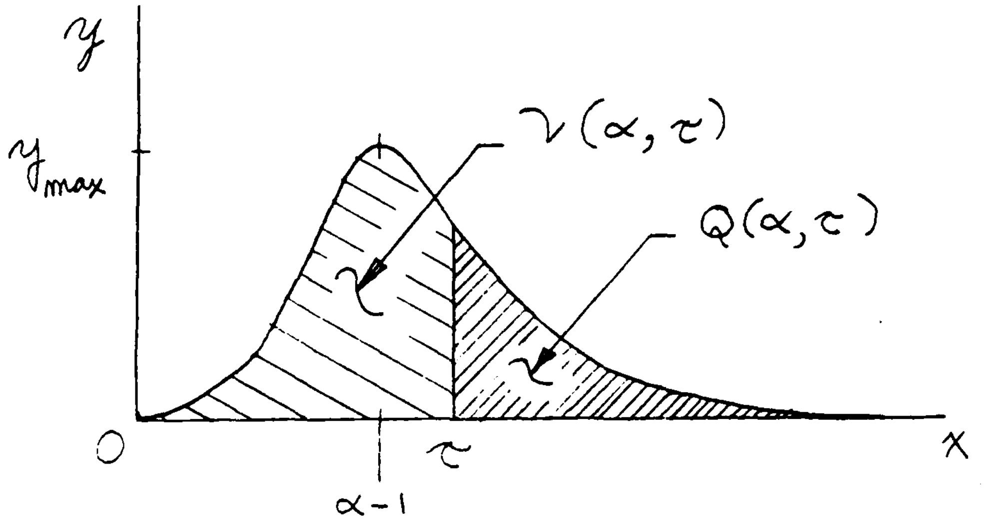


FIGURE 2 PICTORIAL REPRESENTATION OF THE INCOMPLETE GAMMA FUNCTIONS

EXAMPLE

FIND: $\gamma(1.5,0.9)$, $Q(1.5,0.9)$

SOLUTION: $\alpha = 1.5$; $\tau = 0.9$

$\Gamma(1.5) = 0.8862269255$ from Table I (Table 6.1 page 268 [1])

$\chi^2 = 2\tau = 1.8$; $\nu = 2\alpha = 3$

In Table 26.7 page 978 [1]

$Q(\chi^2|\nu) = 0.61493$

$P(\chi^2|\nu) = 1 - Q(\chi^2|\nu) = 0.38507$

$\gamma(1.5,0.9) = \Gamma(1.5)\, P(\chi^2|\nu) = 0.34126$

$Q(1.5,0.9) = \Gamma(1.5)\, Q(\chi^2|\nu) = 0.54497$

As a check

$\gamma(1.5,0.9) + Q(1.5,0.9) = 0.88623$ to five places $= \Gamma(1.5)$

Interpolation formulas are shown on the bottom of p. 981 [1] (Table III). One formula is for interpolation on χ^2 alone. The Double Entry Enterpolation formula is for interpolation on both χ^2 and ν.

where $\phi = \frac{1}{2}(\chi^2 - \chi_o^2)$

$w = \nu - \nu_o > 0$

For interpolation on ν alone, the Double Entry formula can be modified by letting $\phi = 0$ (i.e. $\chi^2 = \chi_o^2$) as follows:

$$Q(\chi^2|\nu) = Q(\chi_o^2|\nu_o - 1)\left[\tfrac{1}{2}w^2 - \frac{w}{2}\right] + Q(\chi_o^2|\nu_o)\left[1 - w^2\right] + Q(\chi_o^2|\nu_o + 1)\left[\tfrac{1}{2}w^2 + \frac{w}{2}\right]$$

EXAMPLE:

FIND: $\gamma(5.64, 8)$, $Q(5.64, 8)$

SOLUTION:

$\alpha = 5.64$; $\tau = 8$

$\Gamma(5.64) = 4.64 \times 3.64 \times 2.64 \times 1.64 \times \Gamma(1.64)$

$\Gamma(1.64) = 0.8986420302$ from Table I (Table 6.1 page 269 [1])

$\Gamma(5.64) = 65.71338911$

$\chi_o^2 = \chi^2 = 2\tau = 16$; $\nu = 2\alpha = 11.28$

In this example

$\nu_o = 11$; $\nu_o - 1 = 10$; $\nu_o + 1 = 12$

$w = \nu - \nu_o = 11.28 - 11 = 0.28$

Thus

$$Q(\chi^2|\nu) = (0.09963)\left[\frac{.28^2}{2} - \frac{.28}{2}\right]$$

$$+ (0.14113)\left[1 - .28^2\right]$$

$$+ (0.19124)\left[\frac{.28^2}{2} + \frac{.28}{2}\right]$$

$$= -\ 0.01004 + 0.13006 + 0.03427$$

$$Q(\chi^2|\nu) = 0.15429$$

$$P(\chi^2|\nu) = 1 - Q\ (\chi^2|\nu) = 0.84571$$

$$\gamma(5.64,\ 8) = \Gamma(5.64)\ P(\chi^2|\nu) = 55.574$$

$$Q(5.64,\ 8) = (5.64)\ Q(\chi^2|\nu) = 10.139$$

As a check

$\gamma(5.64, 8) + Q(5.64, 8) = 65.713$ to five places $= \Gamma(5.64)$

EXAMPLE

FIND: $\gamma(4.3, 3.77)$, $Q\ (4.3, 3.77)$

SOLUTION:

$$\alpha = 4.3 \quad ; \quad \tau = 3.77$$

$$\Gamma\ (4.3) = 3.3 \times 2.3 \times 1.3 \times \Gamma(1.3)$$

$$\Gamma\ (4.3) = 8.855343359$$

$$\chi^2 = 2\tau = 7.54\ ;\ \nu = 2\alpha = 8.6$$

For this example

$$\chi^2 = 7.54\ ;\ \chi_o^2 = 7.4$$

$$\phi = \tfrac{1}{2}\ (7.54 - 7.4) = 0.07$$

$$\nu = 8.6\ ;\ \nu_o = 8\ ;\ \nu_o + 1 = 9$$

$$\nu_o - 4 = 4$$

$$\nu_o - 2 = 6$$

$$\nu_o - 1 = 7$$

EXAMPLE (Cont'd)

$$w = 8.6 - 8 = 0.6$$

The Double Entry Interpolation expression on page 981 [1] (Table III) will be used.

$$Q(\chi_o^2|\nu) = Q(\chi_o^2 \,|\nu_o -4)\left[\frac{1}{2} \quad \phi^2\right]$$
$$+Q(\chi_o^2|\nu_o -2)\left[\phi - \phi^2 - w\,\phi\right]$$
$$+Q(\chi_o|\nu_o -1)\left[\frac{1}{2}w^2 - \frac{1}{2}w + w\phi\right]$$
$$+Q(\chi_o|\nu_o)\left[1 - w^2 - \phi + \frac{1}{2}\phi^2 + w\,\phi\right]$$
$$+Q(\chi_o|\nu_o + 1)\left[\frac{1}{2}w^2 + \frac{1}{2}w - w\,\phi\right]$$

$$Q(\chi^2|\nu) = (0.11620)\left[\frac{1}{2} \quad (.07)^2\right]$$
$$+ (0.28543)\left[.07 - .07^2 - (.6)(.07)\right]$$
$$+ (0.38845)\left[\frac{1}{2}(.6)^2 - \frac{1}{2}(.6) + (.6)(.07)\right]$$
$$+ (0.49415)\left[1 - .6^2 - .07 + \frac{1}{2}(.07)^2 + (.6)(.07)\right]$$
$$+ (0.59555)\left[\frac{1}{2}(.6)^2 + \frac{1}{2}(.6) - (.6)(.07)\right]$$

$$Q(\chi^2|\nu) = 0.00028469 + 0.006593433$$
$$-0.0302991 + 0.30363046 + 0.2608509$$
$$= 0.5410603905$$

$Q(4.3,\ 3.77) = \Gamma(4.3)\ Q(\chi^2|\nu)$

$= 4.7913$ to five places   4.791276 to seven places

$\gamma\ (4.3,\ 3.77) = \Gamma(4.3)\left[1 - Q(\chi^2|\nu)\right]$

$= 4.0641$ to five places   4.064068 to seven places

8.85534 to six places

TABLE III TABULATION OF $Q(\chi^2|\nu)$

Table 26.7 PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION, CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

ν	χ^2 0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010
	m 0.0005	0.0010	0.0015	0.0020	0.0025	0.0030	0.0035	0.0040	0.0045	0.0050
1	0.97477	0.96433	0.95632	0.94957	0.94363	0.93826	0.93332	0.92873	0.92442	0.92034
2	0.99950	0.99900	0.99850	0.99800	0.99750	0.99700	0.99651	0.99601	0.99551	0.99501
3	0.99999	0.99998	0.99996	0.99993	0.99991	0.99988	0.99984	0.99981	0.99977	0.99973
4							0.99999	0.99999	0.99999	0.99999

ν	χ^2 0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
	m 0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040	0.045	0.050
1	0.92034	0.88754	0.86249	0.84148	0.82306	0.80650	0.79134	0.77730	0.76418	0.75183
2	0.99501	0.99005	0.98511	0.98020	0.97531	0.97045	0.96561	0.96079	0.95600	0.95123
3	0.99973	0.99925	0.99863	0.99790	0.99707	0.99616	0.99518	0.99412	0.99301	0.99184
4	0.99999	0.99995	0.99989	0.99980	0.99969	0.99956	0.99940	0.99922	0.99902	0.99879
5			0.99999	0.99998	0.99997	0.99995	0.99993	0.99991	0.99987	0.99984
6							0.99999	0.99999	0.99999	0.99998

ν	χ^2 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	m 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
1	0.75183	0.65472	0.58388	0.52709	0.47950	0.43858	0.40278	0.37109	0.34278	0.31731
2	0.95123	0.90484	0.86071	0.81873	0.77880	0.74082	0.70469	0.67032	0.63763	0.60653
3	0.99184	0.97759	0.96003	0.94024	0.91889	0.89643	0.87320	0.84947	0.82543	0.80125
4	0.99879	0.99532	0.98981	0.98248	0.97350	0.96306	0.95133	0.93845	0.92456	0.90980
5	0.99984	0.99911	0.99764	0.99533	0.99212	0.98800	0.98297	0.97703	0.97022	0.96257
6	0.99998	0.99985	0.99950	0.99885	0.99784	0.99640	0.99449	0.99207	0.98912	0.98561
7		0.99997	0.99990	0.99974	0.99945	0.99899	0.99834	0.99744	0.99628	0.99483
8			0.99998	0.99994	0.99987	0.99973	0.99953	0.99922	0.99880	0.99825
9				0.99999	0.99997	0.99993	0.99987	0.99978	0.99964	0.99944
10					0.99999	0.99998	0.99997	0.99994	0.99989	0.99983
11							0.99999	0.99998	0.99997	0.99995
12									0.99999	0.99999

ν	χ^2 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
	m 0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
1	0.29427	0.27332	0.25421	0.23672	0.22067	0.20590	0.19229	0.17971	0.16808	0.15730
2	0.57695	0.54881	0.52205	0.49659	0.47237	0.44933	0.42741	0.40657	0.38674	0.36788
3	0.77707	0.75300	0.72913	0.70553	0.68227	0.65939	0.63693	0.61493	0.59342	0.57241
4	0.89427	0.87810	0.86138	0.84420	0.82664	0.80879	0.79072	0.77248	0.75414	0.73576
5	0.95410	0.94488	0.93493	0.92431	0.91307	0.90125	0.88890	0.87607	0.86280	0.84915
6	0.98154	0.97689	0.97166	0.96586	0.95949	0.95258	0.94512	0.93714	0.92866	0.91970
7	0.99305	0.99093	0.98844	0.98557	0.98231	0.97864	0.97457	0.97008	0.96517	0.95984
8	0.99753	0.99664	0.99555	0.99425	0.99271	0.99092	0.98887	0.98654	0.98393	0.98101
9	0.99917	0.99882	0.99838	0.99782	0.99715	0.99633	0.99537	0.99425	0.99295	0.99147
10	0.99973	0.99961	0.99944	0.99921	0.99894	0.99859	0.99817	0.99766	0.99705	0.99634
11	0.99992	0.99987	0.99981	0.99973	0.99962	0.99948	0.99930	0.99908	0.99882	0.99850
12	0.99998	0.99996	0.99994	0.99991	0.99987	0.99982	0.99975	0.99966	0.99954	0.99941
13	0.99999	0.99999	0.99998	0.99997	0.99996	0.99994	0.99991	0.99988	0.99983	0.99977
14			0.99999	0.99999	0.99999	0.99998	0.99997	0.99996	0.99994	0.99992
15						0.99999	0.99999	0.99999	0.99998	0.99997
16									0.99999	0.99999

$$Q(\chi^2|\nu)=1-P(\chi^2|\nu)=\left[2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\chi^2}^{\infty}e^{-\frac{t}{2}}t^{\frac{\nu}{2}-1}dt=\left[\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\frac{1}{2}\chi^2}^{\infty}e^{-t}t^{\frac{\nu}{2}-1}dt=\sum_{j=0}^{c-1}e^{-m}m^j/j!\quad(\nu \text{ even}, c=\tfrac{1}{2}\nu, m=\tfrac{1}{2}\chi^2)$$

Compiled from E. S. Pearson and H. O. Hartley (editors), Biometrika tables for statisticians, vol. 1. Cambridge Univ. Press, Cambridge, England, 1954 (with permission).

TABLE III (Cont'd)

PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION Table 26.7
CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

χ^2	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
ν \ m	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1	0.13801	0.12134	0.10686	0.09426	0.08327	0.07364	0.06520	0.05778	0.05125	0.04550
2	0.33287	0.30119	0.27253	0.24660	0.22313	0.20190	0.18268	0.16530	0.14957	0.13534
3	0.53195	0.49363	0.45749	0.42350	0.39163	0.36181	0.33397	0.30802	0.28389	0.26146
4	0.69903	0.66263	0.62682	0.59183	0.55783	0.52493	0.49325	0.46284	0.43375	0.40601
5	0.82084	0.79147	0.76137	0.73079	0.69999	0.66918	0.63857	0.60831	0.57856	0.54942
6	0.90042	0.87949	0.85711	0.83350	0.80885	0.78336	0.75722	0.73062	0.70372	0.67668
7	0.94795	0.93444	0.91938	0.90287	0.88500	0.86590	0.84570	0.82452	0.80250	0.77978
8	0.97426	0.96623	0.95691	0.94628	0.93436	0.92119	0.90681	0.89129	0.87470	0.85712
9	0.98790	0.98345	0.97807	0.97170	0.96430	0.95583	0.94631	0.93572	0.92408	0.91141
10	0.99457	0.99225	0.98934	0.98575	0.98142	0.97632	0.97039	0.96359	0.95592	0.94735
11	0.99766	0.99652	0.99503	0.99311	0.99073	0.98781	0.98431	0.98019	0.97541	0.96992
12	0.99903	0.99850	0.99777	0.99680	0.99554	0.99396	0.99200	0.98962	0.98678	0.98344
13	0.99961	0.99938	0.99903	0.99856	0.99793	0.99711	0.99606	0.99475	0.99314	0.99119
14	0.99985	0.99975	0.99960	0.99938	0.99907	0.99866	0.99813	0.99743	0.99655	0.99547
15	0.99994	0.99990	0.99984	0.99974	0.99960	0.99940	0.99913	0.99878	0.99832	0.99774
16	0.99998	0.99996	0.99994	0.99989	0.99983	0.99974	0.99961	0.99944	0.99921	0.99890
17	0.99999	0.99999	0.99998	0.99996	0.99993	0.99989	0.99983	0.99975	0.99964	0.99948
18			0.99999	0.99998	0.99997	0.99995	0.99993	0.99989	0.99984	0.99976
19				0.99999	0.99999	0.99998	0.99997	0.99995	0.99993	0.99989
20						0.99999	0.99999	0.99998	0.99997	0.99995
21								0.99999	0.99999	0.99998
22										0.99999

χ^2	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
ν \ m	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
1	0.04042	0.03594	0.03197	0.02846	0.02535	0.02259	0.02014	0.01796	0.01603	0.01431
2	0.12246	0.11080	0.10026	0.09072	0.08209	0.07427	0.06721	0.06081	0.05502	0.04979
3	0.24066	0.22139	0.20354	0.18704	0.17180	0.15772	0.14474	0.13278	0.12176	0.11161
4	0.37962	0.35457	0.33085	0.30844	0.28730	0.26739	0.24866	0.23108	0.21459	0.19915
5	0.52099	0.49337	0.46662	0.44077	0.41588	0.39196	0.36904	0.34711	0.32617	0.30622
6	0.64963	0.62271	0.59604	0.56971	0.54381	0.51843	0.49363	0.46945	0.44596	0.42319
7	0.75647	0.73272	0.70864	0.68435	0.65996	0.63557	0.61127	0.58715	0.56329	0.53975
8	0.83864	0.81935	0.79935	0.77872	0.75758	0.73600	0.71409	0.69194	0.66962	0.64723
9	0.89776	0.88317	0.86769	0.85138	0.83431	0.81654	0.79814	0.77919	0.75976	0.73992
10	0.93787	0.92750	0.91625	0.90413	0.89118	0.87742	0.86291	0.84768	0.83178	0.81526
11	0.96370	0.95672	0.94898	0.94046	0.93117	0.92109	0.91026	0.89868	0.88637	0.87337
12	0.97955	0.97509	0.97002	0.96433	0.95798	0.95096	0.94327	0.93489	0.92583	0.91608
13	0.98887	0.98614	0.98298	0.97934	0.97519	0.97052	0.96530	0.95951	0.95313	0.94615
14	0.99414	0.99254	0.99064	0.98841	0.98581	0.98283	0.97943	0.97559	0.97128	0.96649
15	0.99701	0.99610	0.99501	0.99369	0.99213	0.99029	0.98816	0.98571	0.98291	0.97975
16	0.99851	0.99802	0.99741	0.99666	0.99575	0.99467	0.99338	0.99187	0.99012	0.98810
17	0.99928	0.99902	0.99869	0.99828	0.99777	0.99715	0.99639	0.99550	0.99443	0.99319
18	0.99966	0.99953	0.99936	0.99914	0.99886	0.99851	0.99809	0.99757	0.99694	0.99620
19	0.99985	0.99978	0.99969	0.99958	0.99943	0.99924	0.99901	0.99872	0.99836	0.99793
20	0.99993	0.99990	0.99986	0.99980	0.99972	0.99962	0.99950	0.99934	0.99914	0.99890
21	0.99997	0.99995	0.99993	0.99991	0.99987	0.99982	0.99975	0.99967	0.99956	0.99943
22	0.99999	0.99998	0.99997	0.99996	0.99994	0.99991	0.99988	0.99984	0.99978	0.99971
23	0.99999	0.99999	0.99999	0.99998	0.99997	0.99996	0.99994	0.99992	0.99989	0.99986
24			0.99999	0.99999	0.99999	0.99998	0.99997	0.99996	0.99995	0.99993
25					0.99999	0.99999	0.99999	0.99998	0.99998	0.99997
26								0.99999	0.99999	0.99998
27									0.99999	0.99999

$\phi=\frac{1}{2}(\chi^2-\chi_0^2)\qquad w=\nu-\nu_0>0$

Interpolation on χ^2

$$Q(\chi^2|\nu)=Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right]+Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2\right]+Q(\chi_0^2|\nu_0)\left[1-\phi+\tfrac{1}{2}\phi^2\right]$$

Double Entry Interpolation

$$Q(\chi^2|\nu)=Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right]+Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2-w\phi\right]+Q(\chi_0^2|\nu_0-1)\left[\tfrac{1}{2}w^2-\tfrac{1}{2}w+w\phi\right]$$
$$+Q(\chi_0^2|\nu_0)\left[1-w^2-\phi+\tfrac{1}{2}\phi^2+w\phi\right]+Q(\chi_0^2|\nu_0+1)\left[\tfrac{1}{2}w^2+\tfrac{1}{2}w-w\phi\right]$$

TABLE III (Cont'd)

Table 26.7 PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

ν	χ^2 6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
	m 3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
1	0.01278	0.01141	0.01020	0.00912	0.00815	0.00729	0.00652	0.00584	0.00522	0.00468
2	0.04505	0.04076	0.03688	0.03337	0.03020	0.02732	0.02472	0.02237	0.02024	0.01832
3	0.10228	0.09369	0.08580	0.07855	0.07190	0.06579	0.06018	0.05504	0.05033	0.04601
4	0.18470	0.17120	0.15860	0.14684	0.13589	0.12569	0.11620	0.10738	0.09919	0.09158
5	0.28724	0.26922	0.25213	0.23595	0.22064	0.20619	0.19255	0.17970	0.16761	0.15624
6	0.40116	0.37990	0.35943	0.33974	0.32085	0.30275	0.28543	0.26890	0.25313	0.23810
7	0.51660	0.49390	0.47168	0.45000	0.42888	0.40836	0.38845	0.36918	0.35056	0.33259
8	0.62484	0.60252	0.58034	0.55836	0.53663	0.51522	0.49415	0.47349	0.45325	0.43347
9	0.71975	0.69931	0.67869	0.65793	0.63712	0.61631	0.59555	0.57490	0.55442	0.53415
10	0.79819	0.78061	0.76259	0.74418	0.72544	0.70644	0.68722	0.66784	0.64837	0.62884
11	0.85969	0.84539	0.83049	0.81504	0.79908	0.78266	0.76583	0.74862	0.73110	0.71330
12	0.90567	0.89459	0.88288	0.87054	0.85761	0.84412	0.83009	0.81556	0.80056	0.78513
13	0.93857	0.93038	0.92157	0.91216	0.90215	0.89155	0.88038	0.86865	0.85638	0.84360
14	0.96120	0.95538	0.94903	0.94215	0.93471	0.92673	0.91819	0.90911	0.89948	0.88933
15	0.97619	0.97222	0.96782	0.96296	0.95765	0.95186	0.94559	0.93882	0.93155	0.92378
16	0.98579	0.98317	0.98022	0.97693	0.97326	0.96921	0.96476	0.95989	0.95460	0.94887
17	0.99174	0.99007	0.98816	0.98599	0.98355	0.98081	0.97775	0.97437	0.97064	0.96655
18	0.99532	0.99429	0.99309	0.99171	0.99013	0.98833	0.98630	0.98402	0.98147	0.97864
19	0.99741	0.99679	0.99606	0.99521	0.99421	0.99307	0.99176	0.99026	0.98857	0.98667
20	0.99860	0.99824	0.99781	0.99729	0.99669	0.99598	0.99515	0.99420	0.99311	0.99187
21	0.99926	0.99905	0.99880	0.99850	0.99814	0.99771	0.99721	0.99662	0.99594	0.99514
22	0.99962	0.99950	0.99936	0.99919	0.99898	0.99873	0.99843	0.99807	0.99765	0.99716
23	0.99981	0.99974	0.99967	0.99957	0.99945	0.99931	0.99913	0.99892	0.99867	0.99837
24	0.99990	0.99987	0.99983	0.99978	0.99971	0.99963	0.99953	0.99941	0.99926	0.99908
25	0.99995	0.99994	0.99991	0.99989	0.99985	0.99981	0.99975	0.99968	0.99960	0.99949
26	0.99998	0.99997	0.99996	0.99994	0.99992	0.99990	0.99987	0.99983	0.99978	0.99973
27	0.99999	0.99999	0.99998	0.99997	0.99996	0.99995	0.99993	0.99991	0.99989	0.99985
28		0.99999	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996	0.99994	0.99992
29				0.99999	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996
30						0.99999	0.99999	0.99999	0.99999	0.99998

ν	χ^2 8.2	8.4	8.6	8.8	9.0	9.2	9.4	9.6	9.8	10.0
	m 4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
1	0.00419	0.00375	0.00336	0.00301	0.00270	0.00242	0.00217	0.00195	0.00175	0.00157
2	0.01657	0.01500	0.01357	0.01228	0.01111	0.01005	0.00910	0.00823	0.00745	0.00674
3	0.04205	0.03843	0.03511	0.03207	0.02929	0.02675	0.02442	0.02229	0.02034	0.01857
4	0.08452	0.07798	0.07191	0.06630	0.06110	0.05629	0.05184	0.04773	0.04394	0.04043
5	0.14555	0.13553	0.12612	0.11731	0.10906	0.10135	0.09413	0.08740	0.08110	0.07524
6	0.22381	0.21024	0.19736	0.18514	0.17358	0.16264	0.15230	0.14254	0.13333	0.12465
7	0.31529	0.29865	0.28266	0.26734	0.25266	0.23861	0.22520	0.21240	0.20019	0.18857
8	0.41418	0.39540	0.37715	0.35945	0.34230	0.32571	0.30968	0.29423	0.27935	0.26503
9	0.51412	0.49439	0.47499	0.45594	0.43727	0.41902	0.40120	0.38383	0.36692	0.35049
10	0.60931	0.58983	0.57044	0.55118	0.53210	0.51323	0.49461	0.47626	0.45821	0.44049
11	0.69528	0.67709	0.65876	0.64035	0.62189	0.60344	0.58502	0.56669	0.54846	0.53039
12	0.76931	0.75314	0.73666	0.71991	0.70293	0.68576	0.66844	0.65101	0.63350	0.61596
13	0.83033	0.81660	0.80244	0.78788	0.77294	0.75768	0.74211	0.72627	0.71020	0.69393
14	0.87865	0.86746	0.85579	0.84365	0.83105	0.81803	0.80461	0.79081	0.77666	0.76218
15	0.91551	0.90675	0.89749	0.88774	0.87752	0.86683	0.85569	0.84412	0.83213	0.81974
16	0.94269	0.93606	0.92897	0.92142	0.91341	0.90495	0.89603	0.88667	0.87686	0.86663
17	0.96208	0.95723	0.95198	0.94633	0.94026	0.93378	0.92687	0.91954	0.91179	0.90361
18	0.97551	0.97207	0.96830	0.96420	0.95974	0.95493	0.94974	0.94418	0.93824	0.93191
19	0.98454	0.98217	0.97955	0.97666	0.97348	0.97001	0.96623	0.96213	0.95771	0.95295
20	0.99046	0.98887	0.98709	0.98511	0.98291	0.98047	0.97779	0.97486	0.97166	0.96817
21	0.99424	0.99320	0.99203	0.99070	0.98921	0.98755	0.98570	0.98365	0.98139	0.97891
22	0.99659	0.99593	0.99518	0.99431	0.99333	0.99222	0.99098	0.98958	0.98803	0.98630
23	0.99802	0.99761	0.99714	0.99659	0.99596	0.99524	0.99442	0.99349	0.99245	0.99128
24	0.99888	0.99863	0.99833	0.99799	0.99760	0.99714	0.99661	0.99601	0.99532	0.99455
25	0.99937	0.99922	0.99905	0.99884	0.99860	0.99831	0.99798	0.99760	0.99716	0.99665
26	0.99966	0.99957	0.99947	0.99934	0.99919	0.99902	0.99882	0.99858	0.99830	0.99798
27	0.99981	0.99977	0.99971	0.99963	0.99955	0.99944	0.99932	0.99917	0.99900	0.99880
28	0.99990	0.99987	0.99984	0.99980	0.99975	0.99969	0.99962	0.99953	0.99942	0.99930
29	0.99995	0.99993	0.99991	0.99989	0.99986	0.99983	0.99979	0.99973	0.99967	0.99960
30	0.99997	0.99997	0.99996	0.99994	0.99993	0.99991	0.99988	0.99985	0.99982	0.99977

$$Q(\chi^2|\nu)=1-P(\chi^2|\nu)=\left[2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\chi^2}^{\infty}e^{-\frac{t}{2}}t^{\frac{\nu}{2}-1}dt=\left[\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\frac{1}{2}\chi^2}^{\infty}e^{-t}t^{\frac{\nu}{2}-1}dt=\sum_{j=0}^{c-1}e^{-m}m^j/j!\;(\nu\text{ even},\ c=\tfrac{1}{2}\nu,\ m=\tfrac{1}{2}\chi^2)$$

TABLE III (Cont'd)

Table 26.7

PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

χ^2=	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0	14.5	15.0
ν m=	5.25	5.5	5.75	6.0	6.25	6.5	6.75	7.0	7.25	7.5
1	0.00119	0.00091	0.00070	0.00053	0.00041	0.00031	0.00024	0.00018	0.00014	0.00011
2	0.00525	0.00409	0.00318	0.00248	0.00193	0.00150	0.00117	0.00091	0.00071	0.00055
3	0.01476	0.01173	0.00931	0.00738	0.00585	0.00464	0.00367	0.00291	0.00230	0.00182
4	0.03280	0.02656	0.02148	0.01735	0.01400	0.01128	0.00907	0.00730	0.00586	0.00470
5	0.06225	0.05138	0.04232	0.03479	0.02854	0.02338	0.01912	0.01561	0.01273	0.01036
6	0.10511	0.08838	0.07410	0.06197	0.05170	0.04304	0.03575	0.02964	0.02452	0.02026
7	0.16196	0.13862	0.11825	0.10056	0.08527	0.07211	0.06082	0.05118	0.04297	0.03600
8	0.23167	0.20170	0.17495	0.15120	0.13025	0.11185	0.09577	0.08177	0.06963	0.05915
9	0.31154	0.27571	0.24299	0.21331	0.18657	0.16261	0.14126	0.12233	0.10562	0.09094
10	0.39777	0.35752	0.31991	0.28506	0.25299	0.22367	0.19704	0.17299	0.15138	0.13206
11	0.48605	0.44326	0.40237	0.36364	0.32726	0.29333	0.26190	0.23299	0.20655	0.18250
12	0.57218	0.52892	0.48662	0.44568	0.40640	0.36904	0.33377	0.30071	0.26992	0.24144
13	0.65263	0.61082	0.56901	0.52764	0.48713	0.44781	0.40997	0.37384	0.33960	0.30735
14	0.72479	0.68604	0.64639	0.60630	0.56622	0.52652	0.48759	0.44971	0.41316	0.37815
15	0.78717	0.75259	0.71641	0.67903	0.64086	0.60230	0.56374	0.52553	0.48800	0.45142
16	0.83925	0.80949	0.77762	0.74398	0.70890	0.67276	0.63591	0.59871	0.56152	0.52464
17	0.88135	0.85656	0.82942	0.80014	0.76896	0.73619	0.70212	0.66710	0.63145	0.59548
18	0.91436	0.89436	0.87195	0.84724	0.82038	0.79157	0.76106	0.72909	0.69596	0.66197
19	0.93952	0.92384	0.90587	0.88562	0.86316	0.83857	0.81202	0.78369	0.75380	0.72260
20	0.95817	0.94622	0.93221	0.91608	0.89779	0.87738	0.85492	0.83050	0.80427	0.77641
21	0.97166	0.96279	0.95214	0.93962	0.92513	0.90862	0.89010	0.86960	0.84718	0.82295
22	0.98118	0.97475	0.96686	0.95738	0.94618	0.93316	0.91827	0.90148	0.88279	0.86224
23	0.98773	0.98319	0.97748	0.97047	0.96201	0.95199	0.94030	0.92687	0.91165	0.89463
24	0.99216	0.98901	0.98498	0.97991	0.97367	0.96612	0.95715	0.94665	0.93454	0.92076
25	0.99507	0.99295	0.99015	0.98657	0.98206	0.97650	0.96976	0.96173	0.95230	0.94138
26	0.99696	0.99555	0.99366	0.99117	0.98798	0.98397	0.97902	0.97300	0.96581	0.95733
27	0.99815	0.99724	0.99598	0.99429	0.99208	0.98925	0.98567	0.98125	0.97588	0.96943
28	0.99890	0.99831	0.99749	0.99637	0.99487	0.99290	0.99037	0.98719	0.98324	0.97844
29	0.99935	0.99893	0.99846	0.99773	0.99672	0.99538	0.99363	0.99138	0.98854	0.98502
30	0.99963	0.99940	0.99907	0.99860	0.99794	0.99704	0.99585	0.99428	0.99227	0.98974

χ^2=	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5	20.0
ν m=	7.75	8.0	8.25	8.5	8.75	9.0	9.25	9.5	9.75	10.0
1	0.00008	0.00006	0.00005	0.00004	0.00003	0.00002	0.00002	0.00001	0.00001	0.00001
2	0.00043	0.00034	0.00026	0.00020	0.00016	0.00012	0.00010	0.00008	0.00006	0.00005
3	0.00144	0.00113	0.00090	0.00071	0.00056	0.00044	0.00035	0.00027	0.00022	0.00017
4	0.00377	0.00302	0.00242	0.00193	0.00154	0.00123	0.00099	0.00079	0.00063	0.00050
5	0.00843	0.00684	0.00555	0.00450	0.00364	0.00295	0.00238	0.00192	0.00155	0.00125
6	0.01670	0.01375	0.01131	0.00928	0.00761	0.00623	0.00510	0.00416	0.00340	0.00277
7	0.03010	0.02512	0.02092	0.01740	0.01444	0.01197	0.00991	0.00819	0.00676	0.00557
8	0.05012	0.04238	0.03576	0.03011	0.02530	0.02123	0.01777	0.01486	0.01240	0.01034
9	0.07809	0.06688	0.05715	0.04872	0.04144	0.03517	0.02980	0.02519	0.02126	0.01791
10	0.11487	0.09963	0.08619	0.07436	0.06401	0.05496	0.04709	0.04026	0.03435	0.02925
11	0.16073	0.14113	0.12356	0.10788	0.09393	0.08158	0.07068	0.06109	0.05269	0.04534
12	0.21522	0.19124	0.16939	0.14960	0.13174	0.11569	0.10133	0.08853	0.07716	0.06709
13	0.27719	0.24913	0.22318	0.19930	0.17744	0.15752	0.13944	0.12310	0.10840	0.09521
14	0.34485	0.31337	0.28380	0.25618	0.23051	0.20678	0.18495	0.16495	0.14671	0.13014
15	0.41604	0.38205	0.34962	0.31886	0.28986	0.26267	0.23729	0.21373	0.19196	0.17193
16	0.48837	0.45296	0.41864	0.38560	0.35398	0.32390	0.29544	0.26866	0.24359	0.22022
17	0.55951	0.52383	0.48871	0.45437	0.42102	0.38884	0.35797	0.32853	0.30060	0.27423
18	0.62740	0.59255	0.55770	0.52311	0.48902	0.45565	0.42320	0.39182	0.36166	0.33282
19	0.69033	0.65728	0.62370	0.58987	0.55603	0.52244	0.48931	0.45684	0.42521	0.39458
20	0.74712	0.71662	0.68516	0.65297	0.62031	0.58741	0.55451	0.52183	0.48957	0.45793
21	0.79705	0.76965	0.74093	0.71111	0.68039	0.64900	0.61718	0.58514	0.55310	0.52126
22	0.83990	0.81589	0.79032	0.76336	0.73519	0.70599	0.67597	0.64533	0.61428	0.58304
23	0.87582	0.85527	0.83304	0.80925	0.78402	0.75749	0.72983	0.70122	0.67185	0.64191
24	0.90527	0.88808	0.86919	0.84866	0.82657	0.80301	0.77810	0.75199	0.72483	0.69678
25	0.92891	0.91483	0.89912	0.88179	0.86287	0.84239	0.82044	0.79712	0.77254	0.74683
26	0.94749	0.93620	0.92341	0.90908	0.89320	0.87577	0.85683	0.83643	0.81464	0.79156
27	0.96182	0.95295	0.94274	0.93112	0.91806	0.90352	0.88750	0.87000	0.85107	0.83076
28	0.97266	0.96582	0.95782	0.94859	0.93805	0.92615	0.91285	0.89814	0.88200	0.86446
29	0.98071	0.97554	0.96939	0.96218	0.95383	0.94427	0.93344	0.92129	0.90779	0.89293
30	0.98659	0.98274	0.97810	0.97258	0.96608	0.95853	0.94986	0.94001	0.92891	0.91654

$\phi = \frac{1}{2}(\chi^2 - \chi_0^2) \qquad w = \nu - \nu_0 \geq 0$

Interpolation on χ^2

$$Q(\chi^2|\nu) = Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right] + Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2\right] + Q(\chi_0^2|\nu_0)\left[1-\phi+\tfrac{1}{2}\phi^2\right]$$

Double Entry Interpolation

$$Q(\chi^2|\nu) = Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right] - Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2-w\phi\right] + Q(\chi_0^2|\nu_0-1)\left[\tfrac{1}{2}w^2-\tfrac{1}{2}w+w\phi\right]$$
$$+ Q(\chi_0^2|\nu_0)\left[1-w^2-\phi+\tfrac{1}{2}\phi^2+w\phi\right] + Q(\chi_0^2|\nu_0+1)\left[\tfrac{1}{2}w^2+\tfrac{1}{2}w-w\phi\right]$$

Table 26.7 PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION, CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

ν	χ^2 6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0
	m 3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
1	0.01278	0.01141	0.01020	0.00912	0.00815	0.00729	0.00652	0.00584	0.00522	0.00468
2	0.04505	0.04076	0.03688	0.03337	0.03020	0.02732	0.02472	0.02237	0.02024	0.01832
3	0.10228	0.09369	0.08580	0.07855	0.07190	0.06579	0.06018	0.05504	0.05033	0.04601
4	0.18470	0.17120	0.15860	0.14684	0.13589	0.12569	0.11620	0.10738	0.09919	0.09158
5	0.28724	0.26922	0.25213	0.23595	0.22064	0.20619	0.19255	0.17970	0.16761	0.15624
6	0.40116	0.37990	0.35943	0.33974	0.32085	0.30275	0.28543	0.26890	0.25313	0.23810
7	0.51660	0.49390	0.47168	0.45000	0.42888	0.40836	0.38845	0.36918	0.35056	0.33259
8	0.62484	0.60252	0.58034	0.55836	0.53663	0.51522	0.49415	0.47349	0.45325	0.43347
9	0.71975	0.69931	0.67869	0.65793	0.63712	0.61631	0.59555	0.57490	0.55442	0.53415
10	0.79819	0.78061	0.76259	0.74418	0.72544	0.70644	0.68722	0.66784	0.64837	0.62884
11	0.85969	0.84539	0.83049	0.81504	0.79908	0.78266	0.76583	0.74862	0.73110	0.71330
12	0.90567	0.89459	0.88288	0.87054	0.85761	0.84412	0.83009	0.81556	0.80056	0.78513
13	0.93857	0.93038	0.92157	0.91216	0.90215	0.89155	0.88038	0.86865	0.85638	0.84360
14	0.96120	0.95538	0.94903	0.94215	0.93471	0.92673	0.91819	0.90911	0.89948	0.88933
15	0.97619	0.97222	0.96782	0.96296	0.95765	0.95186	0.94559	0.93882	0.93155	0.92378
16	0.98579	0.98317	0.98022	0.97693	0.97326	0.96921	0.96476	0.95989	0.95460	0.94887
17	0.99174	0.99007	0.98816	0.98599	0.98355	0.98081	0.97775	0.97437	0.97064	0.96655
18	0.99532	0.99429	0.99309	0.99171	0.99013	0.98833	0.98630	0.98402	0.98147	0.97864
19	0.99741	0.99679	0.99606	0.99521	0.99421	0.99307	0.99176	0.99026	0.98857	0.98667
20	0.99860	0.99824	0.99781	0.99729	0.99669	0.99598	0.99515	0.99420	0.99311	0.99187
21	0.99926	0.99905	0.99880	0.99850	0.99814	0.99771	0.99721	0.99662	0.99594	0.99514
22	0.99962	0.99950	0.99936	0.99919	0.99898	0.99873	0.99843	0.99807	0.99765	0.99716
23	0.99981	0.99974	0.99967	0.99957	0.99945	0.99931	0.99913	0.99892	0.99867	0.99837
24	0.99990	0.99987	0.99983	0.99978	0.99971	0.99963	0.99953	0.99941	0.99926	0.99908
25	0.99995	0.99994	0.99991	0.99989	0.99985	0.99981	0.99975	0.99968	0.99960	0.99949
26	0.99998	0.99997	0.99996	0.99994	0.99992	0.99990	0.99987	0.99983	0.99978	0.99973
27	0.99999	0.99999	0.99998	0.99997	0.99996	0.99995	0.99993	0.99991	0.99989	0.99985
28		0.99999	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996	0.99994	0.99992
29				0.99999	0.99999	0.99999	0.99998	0.99998	0.99997	0.99996
30						0.99999	0.99999	0.99999	0.99999	0.99998

ν	χ^2 8.2	8.4	8.6	8.8	9.0	9.2	9.4	9.6	9.8	10.0
	m 4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
1	0.00419	0.00375	0.00336	0.00301	0.00270	0.00242	0.00217	0.00195	0.00175	0.00157
2	0.01657	0.01500	0.01357	0.01228	0.01111	0.01005	0.00910	0.00823	0.00745	0.00674
3	0.04205	0.03843	0.03511	0.03207	0.02929	0.02675	0.02442	0.02229	0.02034	0.01857
4	0.08452	0.07798	0.07191	0.06630	0.06110	0.05629	0.05184	0.04773	0.04394	0.04043
5	0.14555	0.13553	0.12612	0.11731	0.10906	0.10135	0.09413	0.08740	0.08110	0.07524
6	0.22381	0.21024	0.19736	0.18514	0.17358	0.16264	0.15230	0.14254	0.13333	0.12465
7	0.31529	0.29865	0.28266	0.26734	0.25266	0.23861	0.22520	0.21240	0.20019	0.18857
8	0.41418	0.39540	0.37715	0.35945	0.34230	0.32571	0.30968	0.29423	0.27935	0.26503
9	0.51412	0.49439	0.47499	0.45594	0.43727	0.41902	0.40120	0.38383	0.36692	0.35049
10	0.60931	0.58983	0.57044	0.55118	0.53210	0.51323	0.49461	0.47626	0.45821	0.44049
11	0.69528	0.67709	0.65876	0.64035	0.62189	0.60344	0.58502	0.56669	0.54846	0.53039
12	0.76931	0.75314	0.73666	0.71991	0.70293	0.68576	0.66844	0.65101	0.63350	0.61596
13	0.83033	0.81660	0.80244	0.78788	0.77294	0.75768	0.74211	0.72627	0.71020	0.6[illegible]93
14	0.87865	0.86746	0.85579	0.84365	0.83105	0.81803	0.80461	0.79081	0.77666	0.76[illegible]18
15	0.91551	0.90675	0.89749	0.88774	0.87752	0.86683	0.85569	0.84412	0.83213	0.81974
16	0.94269	0.93606	0.92897	0.92142	0.91341	0.90495	0.89603	0.88667	0.87686	0.86663
17	0.96208	0.95723	0.95198	0.94633	0.94026	0.93378	0.92687	0.91954	0.91179	0.90361
18	0.97551	0.97207	0.96830	0.96420	0.95974	0.95493	0.94974	0.94418	0.93824	0.93191
19	0.98454	0.98217	0.97955	0.97666	0.97348	0.97001	0.96623	0.96213	0.95771	0.95295
20	0.99046	0.98887	0.98709	0.98511	0.98291	0.98047	0.97779	0.97486	0.97166	0.96817
21	0.99424	0.99320	0.99203	0.99070	0.98921	0.98755	0.98570	0.98365	0.98139	0.97891
22	0.99659	0.99593	0.99518	0.99431	0.99333	0.99222	0.99098	0.98958	0.98803	0.98630
23	0.99802	0.99761	0.99714	0.99659	0.99596	0.99524	0.99442	0.99349	0.99245	0.99128
24	0.99888	0.99863	0.99833	0.99799	0.99760	0.99714	0.99661	0.99601	0.99532	0.99455
25	0.99937	0.99922	0.99905	0.99884	0.99860	0.99831	0.99798	0.99760	0.99716	0.99665
26	0.99966	0.99957	0.99947	0.99934	0.99919	0.99902	0.99882	0.99858	0.99830	0.99798
27	0.99981	0.99977	0.99971	0.99963	0.99955	0.99944	0.99932	0.99917	0.99900	0.99880
28	0.99990	0.99987	0.99984	0.99980	0.99975	0.99969	0.99962	0.99953	0.99942	0.99930
29	0.99995	0.99993	0.99991	0.99989	0.99986	0.99983	0.99979	0.99973	0.99967	0.99960
30	0.99997	0.99997	0.99996	0.99994	0.99993	0.99991	0.99988	0.99985	0.99982	0.99977

$$Q(\chi^2|\nu)=1-P(\chi^2|\nu)=\left[2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\chi^2}^{\infty}e^{-\frac{t}{2}}t^{\frac{\nu}{2}-1}dt=\left[\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\frac{1}{2}\chi^2}^{\infty}e^{-t}t^{\frac{\nu}{2}-1}dt=\sum_{j=0}^{c-1}e^{-m}m^j/j!\quad(\nu \text{ even}, c=\tfrac{1}{2}\nu, m=\tfrac{1}{2}\chi^2)$$

Table 26.7

PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

$\chi^2=$	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0	14.5	15.0
ν \ $m=$	5.25	5.5	5.75	6.0	6.25	6.5	6.75	7.0	7.25	7.5
1	0.00119	0.00091	0.00070	0.00053	0.00041	0.00031	0.00024	0.00018	0.00014	0.00011
2	0.00525	0.00409	0.00318	0.00248	0.00193	0.00150	0.00117	0.00091	0.00071	0.00055
3	0.01476	0.01173	0.00931	0.00738	0.00585	0.00464	0.00367	0.00291	0.00230	0.00182
4	0.03280	0.02656	0.02148	0.01735	0.01400	0.01128	0.00907	0.00730	0.00586	0.00470
5	0.06225	0.05138	0.04232	0.03479	0.02854	0.02338	0.01912	0.01561	0.01273	0.01036
6	0.10511	0.08838	0.07410	0.06197	0.05170	0.04304	0.03575	0.02964	0.02452	0.02026
7	0.16196	0.13862	0.11825	0.10056	0.08527	0.07211	0.06082	0.05118	0.04297	0.03600
8	0.23167	0.20170	0.17495	0.15120	0.13025	0.11185	0.09577	0.08177	0.06963	0.05915
9	0.31154	0.27571	0.24299	0.21331	0.18657	0.16261	0.14126	0.12233	0.10562	0.09094
10	0.39777	0.35752	0.31991	0.28506	0.25299	0.22367	0.19704	0.17299	0.15138	0.13206
11	0.48605	0.44326	0.40237	0.36364	0.32726	0.29333	0.26190	0.23299	0.20655	0.18250
12	0.57218	0.52892	0.48662	0.44568	0.40640	0.36904	0.33377	0.30071	0.26992	0.24144
13	0.65263	0.61082	0.56901	0.52764	0.48713	0.44781	0.40997	0.37384	0.33960	0.30735
14	0.72479	0.68604	0.64639	0.60630	0.56622	0.52652	0.48759	0.44971	0.41316	0.37815
15	0.78717	0.75259	0.71641	0.67903	0.64086	0.60230	0.56374	0.52553	0.48800	0.45142
16	0.83925	0.80949	0.77762	0.74398	0.70890	0.67276	0.63591	0.59871	0.56152	0.52464
17	0.88135	0.85656	0.82942	0.80014	0.76896	0.73619	0.70212	0.66710	0.63145	0.59548
18	0.91436	0.89436	0.87195	0.84724	0.82038	0.79157	0.76106	0.72909	0.69596	0.66197
19	0.93952	0.92384	0.90587	0.88562	0.86316	0.83857	0.81202	0.78369	0.75380	0.72260
20	0.95817	0.94622	0.93221	0.91608	0.89779	0.87738	0.85492	0.83050	0.80427	0.77641
21	0.97166	0.96279	0.95214	0.93962	0.92513	0.90862	0.89010	0.86960	0.84718	0.82295
22	0.98118	0.97475	0.96686	0.95738	0.94618	0.93316	0.91827	0.90148	0.88279	0.86224
23	0.98773	0.98319	0.97748	0.97047	0.96201	0.95199	0.94030	0.92687	0.91165	0.89463
24	0.99216	0.98901	0.98498	0.97991	0.97367	0.96612	0.95715	0.94665	0.93454	0.92076
25	0.99507	0.99295	0.99015	0.98657	0.98206	0.97650	0.96976	0.96173	0.95230	0.94138
26	0.99696	0.99555	0.99366	0.99117	0.98798	0.98397	0.97902	0.97300	0.96581	0.95733
27	0.99815	0.99724	0.99598	0.99429	0.99208	0.98925	0.98567	0.98125	0.97588	0.96943
28	0.99890	0.99831	0.99749	0.99637	0.99487	0.99290	0.99037	0.98719	0.98324	0.97844
29	0.99935	0.99899	0.99846	0.99773	0.99672	0.99538	0.99363	0.99138	0.98854	0.98502
30	0.99963	0.99940	0.99907	0.99860	0.99794	0.99704	0.99585	0.99428	0.99227	0.98974

$\chi^2=$	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5	20.0
ν \ $m=$	7.75	8.0	8.25	8.5	8.75	9.0	9.25	9.5	9.75	10.0
1	0.00008	0.00006	0.00005	0.00004	0.00003	0.00002	0.00002	0.00001	0.00001	0.00001
2	0.00043	0.00034	0.00026	0.00020	0.00016	0.00012	0.00010	0.00008	0.00006	0.00005
3	0.00144	0.00113	0.00090	0.00071	0.00056	0.00044	0.00035	0.00027	0.00022	0.00017
4	0.00377	0.00302	0.00242	0.00193	0.00154	0.00123	0.00099	0.00079	0.00063	0.00050
5	0.00843	0.00684	0.00555	0.00450	0.00364	0.00295	0.00238	0.00192	0.00155	0.00125
6	0.01670	0.01375	0.01131	0.00928	0.00761	0.00623	0.00510	0.00416	0.00340	0.00277
7	0.03010	0.02512	0.02092	0.01740	0.01444	0.01197	0.00991	0.00819	0.00676	0.00557
8	0.05012	0.04238	0.03576	0.03011	0.02530	0.02123	0.01777	0.01486	0.01240	0.01034
9	0.07809	0.06688	0.05715	0.04872	0.04144	0.03517	0.02980	0.02519	0.02126	0.01791
10	0.11487	0.09963	0.08619	0.07436	0.06401	0.05496	0.04709	0.04026	0.03435	0.02925
11	0.16073	0.14113	0.12356	0.10788	0.09393	0.08158	0.07068	0.06109	0.05269	0.04534
12	0.21522	0.19124	0.16939	0.14960	0.13174	0.11569	0.10133	0.08853	0.07716	0.06709
13	0.27719	0.24913	0.22318	0.19930	0.17744	0.15752	0.13944	0.12310	0.10840	0.09521
14	0.34485	0.31337	0.28380	0.25618	0.23051	0.20678	0.18495	0.16495	0.14671	0.13014
15	0.41604	0.38205	0.34962	0.31886	0.28986	0.26267	0.23729	0.21373	0.19196	0.17193
16	0.48837	0.45296	0.41864	0.38560	0.35398	0.32390	0.29544	0.26866	0.24359	0.22022
17	0.55951	0.52383	0.48871	0.45437	0.42102	0.38884	0.35797	0.32853	0.30060	0.27423
18	0.62740	0.59255	0.55770	0.52311	0.48902	0.45565	0.42320	0.39182	0.36166	0.33282
19	0.69033	0.65728	0.62370	0.58987	0.55603	0.52244	0.48931	0.45684	0.42521	0.39458
20	0.74712	0.71662	0.68516	0.65297	0.62031	0.58741	0.55451	0.52183	0.48957	0.45793
21	0.79705	0.76965	0.74093	0.71111	0.68039	0.64900	0.61718	0.58514	0.55310	0.52126
22	0.83990	0.81589	0.79032	0.76336	0.73519	0.70599	0.67597	0.64533	0.61428	0.58304
23	0.87582	0.85527	0.83304	0.80925	0.78402	0.75749	0.72983	0.70122	0.67185	0.64191
24	0.90527	0.88808	0.86919	0.84866	0.82657	0.80301	0.77810	0.75199	0.72483	0.69678
25	0.92891	0.91483	0.89912	0.88179	0.86287	0.84239	0.82044	0.79712	0.77254	0.74683
26	0.94749	0.93620	0.92341	0.90908	0.89320	0.87577	0.85683	0.83643	0.81464	0.79156
27	0.96182	0.95295	0.94274	0.93112	0.91806	0.90352	0.88750	0.87000	0.85107	0.83076
28	0.97266	0.96582	0.95782	0.94859	0.93805	0.92615	0.91285	0.89814	0.88200	0.86446
29	0.98071	0.97554	0.96939	0.96218	0.95383	0.94427	0.93344	0.92129	0.90779	0.89293
30	0.98659	0.98274	0.97810	0.97258	0.96608	0.95853	0.94986	0.94001	0.92891	0.91654

$\phi=\frac{1}{2}(\chi^2-\chi_0^2)$ $\qquad w=\nu-\nu_0$

Interpolation on χ^2

$$Q(\chi^2|\nu)\approx Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right]+Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2\right]+Q(\chi_0^2|\nu_0)\left[1-\phi+\tfrac{1}{2}\phi^2\right]$$

Double Entry Interpolation

$$Q(\chi^2|\nu)\approx Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right]+Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2-w\phi\right]+Q(\chi_0^2|\nu_0-1)\left[\tfrac{1}{2}w^2-\tfrac{1}{2}w+w\phi\right]$$
$$+Q(\chi_0^2|\nu_0)\left[1-w^2-\phi+\tfrac{1}{2}\phi^2+w\phi\right]+Q(\chi_0^2|\nu_0+1)\left[\tfrac{1}{2}w^2+\tfrac{1}{2}w-w\phi\right]$$

Table 26.7 PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

χ^2	21	22	23	24	25	26	27	28	29	30
ν / m	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0	14.5	15.0
1	0.00001									
2	0.00003	0.00002	0.00001	0.00001						
3	0.00011	0.00007	0.00004	0.00003	0.00002	0.00001	0.00001			
4	0.00032	0.00020	0.00013	0.00008	0.00005	0.00003	0.00002	0.00001	0.00001	0.00001
5	0.00081	0.00052	0.00034	0.00022	0.00014	0.00009	0.00006	0.00004	0.00002	0.00002
6	0.00184	0.00121	0.00080	0.00052	0.00034	0.00022	0.00015	0.00009	0.00006	0.00004
7	0.00377	0.00254	0.00171	0.00114	0.00076	0.00050	0.00033	0.00022	0.00015	0.00010
8	0.00715	0.00492	0.00336	0.00229	0.00155	0.00105	0.00071	0.00047	0.00032	0.00021
9	0.01265	0.00888	0.00620	0.00430	0.00297	0.00204	0.00140	0.00095	0.00065	0.00044
10	0.02109	0.01511	0.01075	0.00760	0.00535	0.00374	0.00260	0.00181	0.00125	0.00086
11	0.03337	0.02437	0.01768	0.01273	0.00912	0.00649	0.00460	0.00324	0.00227	0.00159
12	0.05038	0.03752	0.02773	0.02034	0.01482	0.01073	0.00773	0.00553	0.00394	0.00279
13	0.07293	0.05536	0.04168	0.03113	0.02308	0.01700	0.01244	0.00905	0.00655	0.00471
14	0.10163	0.07861	0.06027	0.04582	0.03457	0.02589	0.01925	0.01423	0.01045	0.00763
15	0.13683	0.10780	0.08414	0.06509	0.04994	0.03802	0.02874	0.02157	0.01609	0.01192
16	0.17851	0.14319	0.11374	0.08950	0.06982	0.05403	0.04148	0.03162	0.02394	0.01800
17	0.22629	0.18472	0.14925	0.11944	0.09471	0.07446	0.05807	0.04494	0.03453	0.02635
18	0.27941	0.23199	0.19059	0.15503	0.12492	0.09976	0.07900	0.06206	0.04838	0.03745
19	0.33680	0.28426	0.23734	0.19615	0.16054	0.13019	0.10465	0.08343	0.06599	0.05180
20	0.39713	0.34051	0.28880	0.24239	0.20143	0.16581	0.13526	0.10940	0.08776	0.06985
21	0.45894	0.39951	0.34398	0.29306	0.24716	0.20645	0.17085	0.14015	0.11400	0.09199
22	0.52074	0.45989	0.40173	0.34723	0.29707	0.25168	0.21123	0.17568	0.14486	0.11846
23	0.58109	0.52025	0.46077	0.40381	0.35029	0.30087	0.25597	0.21578	0.18031	0.14940
24	0.63873	0.57927	0.51980	0.46160	0.40576	0.35317	0.30445	0.26004	0.22013	0.18475
25	0.69261	0.63574	0.57756	0.51937	0.46237	0.40760	0.35588	0.30785	0.26392	0.22429
26	0.74196	0.68870	0.63295	0.57597	0.51898	0.46311	0.40933	0.35846	0.31108	0.26761
27	0.78629	0.73738	0.68501	0.63032	0.57446	0.51860	0.46379	0.41097	0.36090	0.31415
28	0.82535	0.78129	0.73304	0.68154	0.62784	0.57305	0.51825	0.46445	0.41253	0.36322
29	0.85915	0.82019	0.77654	0.72893	0.67825	0.62549	0.57171	0.51791	0.46507	0.41400
30	0.88789	0.85404	0.81526	0.77203	0.72503	0.67513	0.62327	0.57044	0.51760	0.46565

χ^2	31	32	33	34	35	36	37	38	39	40
ν / m	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5	20.0
5	0.00001	0.00001								
6	0.00003	0.00002	0.00001	0.00001						
7	0.00006	0.00004	0.00003	0.00002	0.00001	0.00001				
8	0.00014	0.00009	0.00006	0.00004	0.00003	0.00002	0.00001	0.00001		
9	0.00030	0.00020	0.00013	0.00009	0.00006	0.00004	0.00003	0.00002	0.00001	0.00001
10	0.00059	0.00040	0.00027	0.00019	0.00012	0.00008	0.00006	0.00004	0.00003	0.00002
11	0.00110	0.00076	0.00053	0.00036	0.00025	0.00017	0.00012	0.00008	0.00005	0.00004
12	0.00197	0.00138	0.00097	0.00068	0.00047	0.00032	0.00022	0.00015	0.00011	0.00007
13	0.00337	0.00240	0.00170	0.00120	0.00085	0.00059	0.00041	0.00029	0.00020	0.00014
14	0.00554	0.00401	0.00288	0.00206	0.00147	0.00104	0.00074	0.00052	0.00036	0.00026
15	0.00878	0.00644	0.00469	0.00341	0.00246	0.00177	0.00127	0.00090	0.00064	0.00045
16	0.01346	0.01000	0.00739	0.00543	0.00397	0.00289	0.00210	0.00151	0.00109	0.00078
17	0.01997	0.01505	0.01127	0.00840	0.00622	0.00459	0.00337	0.00246	0.00179	0.00129
18	0.02879	0.02199	0.01669	0.01260	0.00945	0.00706	0.00524	0.00387	0.00285	0.00209
19	0.04037	0.03125	0.02404	0.01838	0.01397	0.01056	0.00793	0.00593	0.00442	0.00327
20	0.05519	0.04330	0.03374	0.02613	0.02010	0.01538	0.01170	0.00886	0.00667	0.00500
21	0.07366	0.05855	0.04622	0.03624	0.02824	0.02187	0.01683	0.01289	0.00981	0.00744
22	0.09612	0.07740	0.06187	0.04912	0.03875	0.03037	0.02366	0.01832	0.01411	0.01081
23	0.12279	0.10014	0.08107	0.06516	0.05202	0.04125	0.03251	0.02547	0.01984	0.01537
24	0.15378	0.12699	0.10407	0.08467	0.06840	0.05489	0.04376	0.03467	0.02731	0.02139
25	0.18902	0.15801	0.13107	0.10791	0.08820	0.07160	0.05774	0.04626	0.03684	0.02916
26	0.22827	0.19312	0.16210	0.13502	0.11165	0.09167	0.07475	0.06056	0.04875	0.03901
27	0.27114	0.23208	0.19707	0.16605	0.13887	0.11530	0.09507	0.07786	0.06336	0.05124
28	0.31708	0.27451	0.23574	0.20087	0.16987	0.14260	0.11886	0.09840	0.08092	0.06613
29	0.36542	0.31987	0.27774	0.23926	0.20454	0.17356	0.14622	0.12234	0.10166	0.08394
30	0.41541	0.36753	0.32254	0.28083	0.24264	0.20808	0.17714	0.14975	0.12573	0.10486

PROBABILITY INTEGRAL OF χ^2-DISTRIBUTION, INCOMPLETE GAMMA FUNCTION Table 26.7
CUMULATIVE SUMS OF THE POISSON DISTRIBUTION

ν	$\chi^2=42$ $m=21$	44 22	46 23	48 24	50 25	52 26	54 27	56 28	58 29	60 30
10	0.00001									
11	0.00002	0.00001								
12	0.00003	0.00002	0.00001							
13	0.00006	0.00003	0.00001	0.00001						
14	0.00012	0.00006	0.00003	0.00001	0.00001					
15	0.00023	0.00011	0.00005	0.00003	0.00001	0.00001				
16	0.00040	0.00020	0.00010	0.00005	0.00002	0.00001	0.00001			
17	0.00067	0.00034	0.00017	0.00009	0.00004	0.00002	0.00001	0.00001		
18	0.00111	0.00058	0.00030	0.00015	0.00008	0.00004	0.00002	0.00001		
19	0.00177	0.00094	0.00050	0.00026	0.00013	0.00007	0.00003	0.00002	0.00001	
20	0.00277	0.00151	0.00081	0.00043	0.00022	0.00011	0.00006	0.00003	0.00001	0.00001
21	0.00421	0.00234	0.00128	0.00069	0.00036	0.00019	0.00010	0.00005	0.00003	0.00001
22	0.00625	0.00355	0.00198	0.00109	0.00059	0.00031	0.00016	0.00009	0.00004	0.00002
23	0.00908	0.00526	0.00299	0.00167	0.00092	0.00050	0.00027	0.00014	0.00007	0.00004
24	0.01291	0.00763	0.00443	0.00252	0.00142	0.00078	0.00043	0.00023	0.00012	0.00006
25	0.01797	0.01085	0.00642	0.00373	0.00213	0.00120	0.00066	0.00036	0.00020	0.00011
26	0.02455	0.01512	0.00912	0.00543	0.00314	0.00180	0.00102	0.00056	0.00031	0.00017
27	0.03292	0.02068	0.01272	0.00768	0.00455	0.00265	0.00152	0.00086	0.00048	0.00026
28	0.04336	0.02779	0.01743	0.01072	0.00647	0.00384	0.00224	0.00129	0.00073	0.00041
29	0.05616	0.03670	0.02346	0.01470	0.00903	0.00545	0.00324	0.00189	0.00109	0.00062
30	0.07157	0.04769	0.03107	0.01983	0.01240	0.00762	0.00460	0.00273	0.00160	0.00092

ν	$\chi^2=62$ $m=31$	64 32	66 33	68 34	70 35	72 36	74 37	76 38
21	0.00001							
22	0.00001	0.00001						
23	0.00002	0.00001	0.00001					
24	0.00003	0.00002	0.00001					
25	0.00006	0.00003	0.00002	0.00001				
26	0.00009	0.00005	0.00003	0.00001	0.00001			
27	0.00014	0.00008	0.00004	0.00002	0.00001	0.00001		
28	0.00023	0.00012	0.00007	0.00004	0.00002	0.00001	0.00001	
29	0.00035	0.00019	0.00011	0.00006	0.00003	0.00002	0.00001	
30	0.00052	0.00029	0.00016	0.00009	0.00005	0.00003	0.00001	0.00001

$$Q(\chi^2|\nu) = 1 - P(\chi^2|\nu) = \left[2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\chi^2}^{\infty} e^{-\frac{t}{2}} t^{\frac{\nu}{2}-1}\,dt = \left[\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\int_{\frac{1}{2}\chi^2}^{\infty} e^{-t} t^{\frac{\nu}{2}-1}\,dt = \sum_{j=0}^{c-1} e^{-m} m^j / j! \quad (\nu \text{ even}, c=\tfrac{1}{2}\nu, m=\tfrac{1}{2}\chi^2)$$

$$\phi = \tfrac{1}{2}(\chi^2 - \chi_0^2) \qquad w = \nu - \nu_0 > 0$$

Interpolation on χ^2

$$Q(\chi^2|\nu) = Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right] + Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2\right] + Q(\chi_0^2|\nu_0)\left[1-\phi+\tfrac{1}{2}\phi^2\right]$$

Double Entry Interpolation

$$Q(\chi^2|\nu) = Q(\chi_0^2|\nu_0-4)\left[\tfrac{1}{2}\phi^2\right] + Q(\chi_0^2|\nu_0-2)\left[\phi-\phi^2-w\phi\right] + Q(\chi_0^2|\nu_0-1)\left[\tfrac{1}{2}w^2-\tfrac{1}{2}w+w\phi\right]$$
$$+Q(\chi_0^2|\nu_0)\left[1-w^2-\phi+\tfrac{1}{2}\phi^2+w\phi\right] + Q(\chi_0^2|\nu_0+1)\left[\tfrac{1}{2}w^2+\tfrac{1}{2}w-w\phi\right]$$

TI-59 METHOD (ML-09)

The TI-59 Programmable Calculator has a numerical integration program it its Master Library, ML-09. Included in PL-4 are pages 29-31 of the TI-59 User's Manual. The method uses Simpson's Rule. $\gamma(\alpha,\tau)$ and $Q(\alpha,\tau)$ can be computed using ML-09. The integration limits will be different for the two Incomplete Gamma Functions.

	$\gamma(\alpha,\tau)$	$Q(\alpha,\tau)$
Lower Limit X_o	0	τ
Upper Limit X_n	τ	∞*

*A suitably large real number must be found to represent ∞. This would be a value such that a large value would result in a negligible increase in the value of the integral. A value of $X_n > 227$ will cause underflow.

A sufficiently large value for n must be established. The larger the value of n used, the better the accuracy of the integral. The computation time also increases.

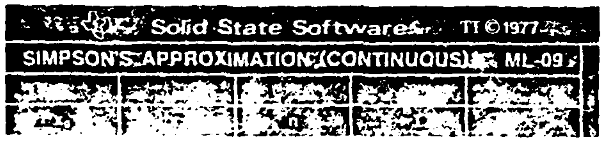
A listing for f(x) is also included.

SIMPSON'S APPROXIMATION (CONTINUOUS)

This program may be used to approximate the integral, I, of a function defined by the user, over an interval x_0 to x_n, using Simpson's Rule.

$$I = \int_{x_0}^{x_n} f(x)dx$$

The function f(x) must be expressed as a sequence of keystrokes in the user program memory.



USER INSTRUCTIONS

STEP	PROCEDURE	ENTER	PRESS	DISPLAY
1	Initialize		[RST]	0.
2	Select learn mode		[LRN]	000 00
3	Use A' as label		[2nd] [Lbl] [2nd] [A']	001 00 002 00
4	Enter f(x) as a series of keystrokes. Do not use [=] or [CLR]. Do not use registers 0-5.			
5	End f(x) with [INV] [SBR]		[INV] [SBR]	xxx 00
6	Leave learn mode		[LRN]	0.
7	Select program		[2nd] [Pgm] 09	
8	Enter lower limit	x_0	[A]	x_0
9	Enter upper limit	x_n	[B]	x_n
10	Enter n (n = 2, 4, 6, · · ·, display flashes if not legal entry)	n	[C]	h
11	Compute integral		[D]	I
12	For a new interval or a new n, repeat Steps 7-11.			

NOTE:
1. Evaluate expressions using parentheses only.
2. Running time is dependent on input data.

PL-4

Example:

Evaluate $\int_0^{\pi/2} \frac{1}{\cos x + 2}\,dx$ using two subintervals.

REF.	ENTER	PRESS	DISPLAY	COMMENTS
		[RST]	0.	
		[LRN]	000 00	
		[2nd] [Lbl] [A']	001 00	
		[2nd] [Lbl]	002 00	
		[2nd]	003 00	
		[( ] [2nd] [cos]	005 00	Key in f(x).
		[+] [2] [ )]	008 00	
		[1/x] [INV] [SBR]	010 00	
		[LRN]	0.	
		[2nd] [Pgm] 09	0.	Select program
1	0	[A]	0.	x_0
		[2nd] [π] [÷]	3.141592654	
2	2	[=] [B]	1.570796327	x_2 (π/2)
3	2	[C]	.7853981634	h
4		[D]	0.604998903	I

OPTIONAL PRINTOUT* REF.	PRINT
1	0. 0.
2	1.570796327 1.570796327
3	2 .7853981634
4	.7853981634 0.604998903

* The printout shown is obtained using the print routine of Program 01.

Register Contents

R_{00}		R_{05}	n	R_{10}		R_{15}	
R_{01}	x_0	R_{06}		R_{11}		R_{16}	
R_{02}	x_n	R_{07}		R_{12}		R_{17}	
R_{03}	h	R_{08}		R_{13}		R_{18}	
R_{04}	I	R_{09}		R_{14}		R_{19}	

Method Used

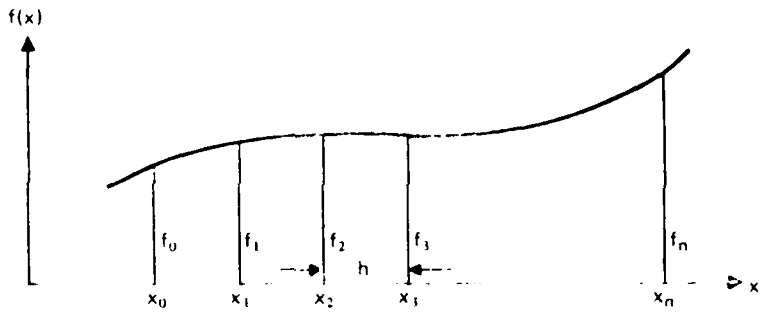
Calculations are based on Simpson's rule:

$$\int_{x_0}^{x_n} f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n)$$

where:

$$h = \frac{x_n - x_0}{n}, \quad x_n > x_0$$

n = number of subintervals = 2, 4, 6, 8, · · ·



LISTING FOR ML-09 f(x)

LRN

LOC	CODE	KEY
000	42	STO
001	00	00
002	53	(
003	43	RCL
004	00	00
005	75	-
006	01	1
007	54	)
008	42	STO
009	08	08
010	91	R/S
011	76	LBL
012	16	A'
013	42	STO
014	06	06
015	53	(
016	43	RCL
017	06	06
018	45	YX
019	43	RCL
020	08	08
021	54	)
022	42	STO
023	07	07
024	53	(
025	53	(
026	43	RCL
027	06	06
028	94	+/-
029	22	INV
030	23	LNX
031	65	X
032	43	RCL
033	07	07
034	54	)
035	92	RTN

LRN

Enter	Press
RST	
α	R/S
2nd Pgm 09	
x_o: lower limit	A
x_n: upper limit	B
n(n = 2, 4, 6 . . .)	C
Compute Integral	D

EXAMPLE

FIND: $\gamma(1.5, 0.9)$ and $Q(1.5, 0.9)$

$\alpha = 1.5$

$x_o = 0$; $x_n = 0.9$

n	$\gamma(1.5, 0.9)$
16	0.3401
32	0.34087
64	0.3411193
128	0.341207
512	0.341249

$\therefore \quad \gamma(1.5, 0.9) \cong 0.341249$

Accuracy could be improved by increasing n still further or by breaking the integral into two integrals with limits from 0 - 0.5 and 0.5 - 0.9 respectively.

$n = 512$

$\gamma_{0-0.5} = 0.1761333892$

$\gamma_{0.5-0.9} = 0.1651191608$

$\gamma(1.5, 0.9) \cong 0.34125255$

n = 128 for each of the following integrals:

$$Q_{0.9-4} = \int_{0.9}^{4} y \, dx = 0.504195085$$

$$Q_{4-8} = 0.0397718$$

$$Q_{8-20} = 0.001005$$

$$Q_{20-40} = 0.0000000094$$

$\therefore \quad Q(1.5, 0.9) \cong 0.54497189$

$\gamma(1.5, 0.9) + Q(1.5, 0.9) = 0.88622444$

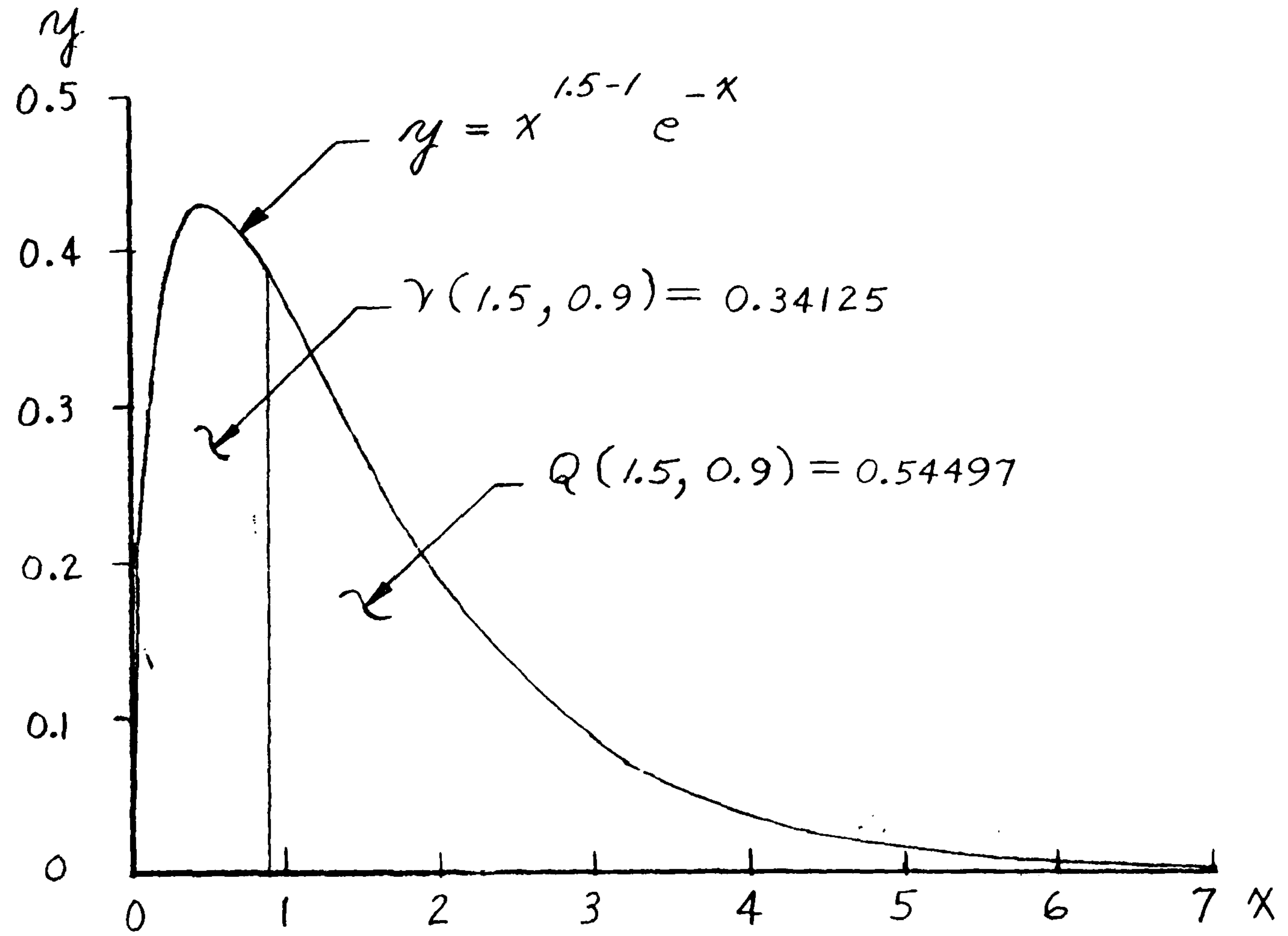


FIGURE 3 GAMMA FUNCTIONS FOR $\alpha = 1.5$; $\tau = 0.9$

EXAMPLE

FIND: $\gamma(5.64, 8)$

$$x_o = 0; \ x_n = 8; \ n = 128$$

$\gamma(5.64, 8) = 55.57279$ (55.5728 to six places)

FIND: $Q(5.64, 8)$

$x_o = 8; \ x_n = \infty$ (say, $x_n = 64$)

$n = 128$

$Q(5.64, 8) = 10.14056$ (10.1406 to six places)

$\gamma(5.64, 8) + Q(5.64, 8) = 65.7134$

FIND: $\Gamma(5.64)$

$x_o = 0 \ ; \ x_n = 64; \ n = 256$

$\Gamma(5.64) \simeq 65.7134$

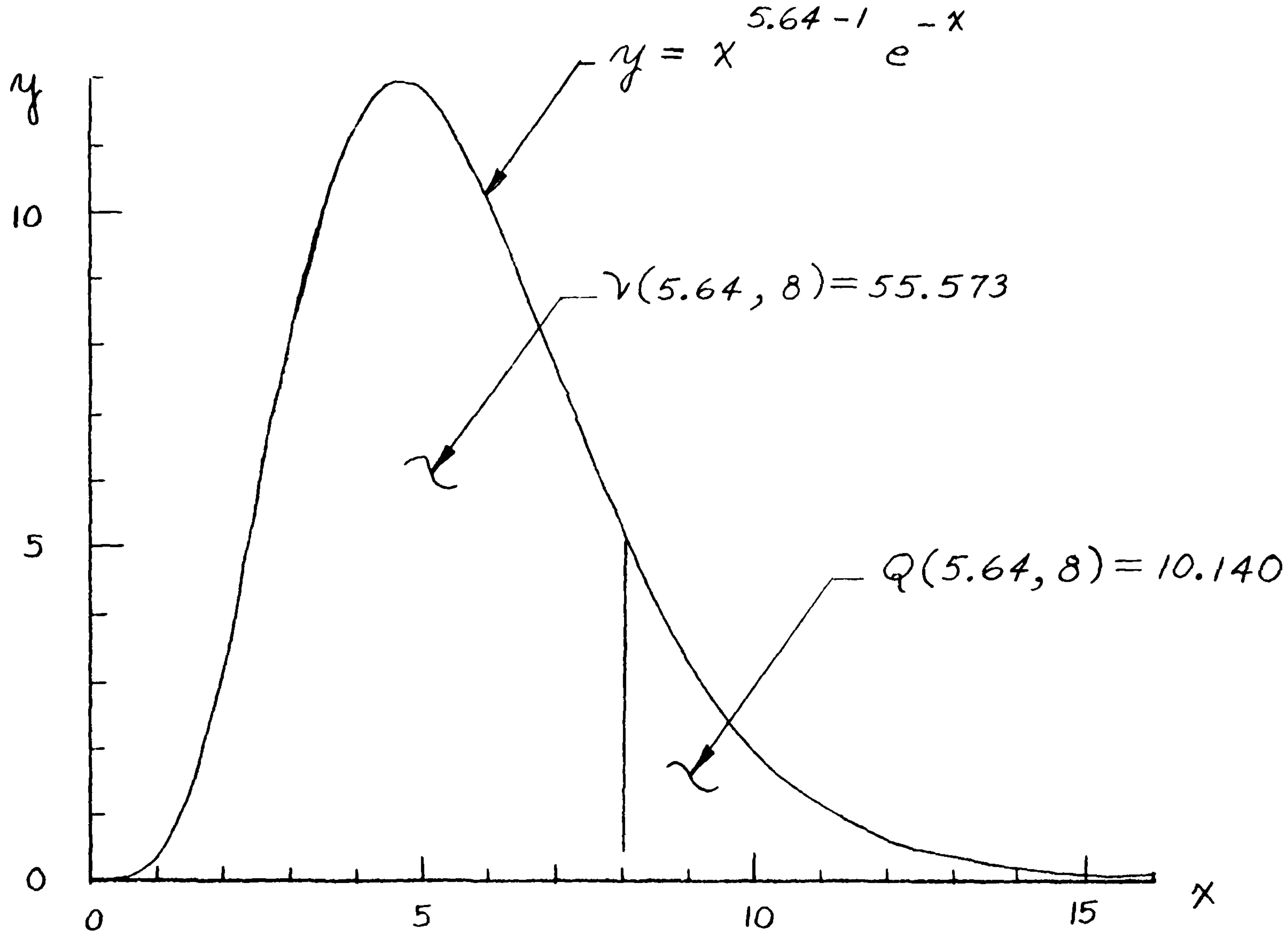


FIGURE 4 GAMMA FUNCTIONS FOR $\alpha = 5.64$; $\tau = 8$

B. TI-59 METHOD (MU-17)

The new Math Utility module can also be used for computing the Gamma Functions. The numerical integration program, MU-17, is much faster in execution time than ML-09 (i.e. PL-4) and simpler to use. This method uses the systematic procedure called Romberg Integration. The subroutine for f(x) must be entered (see PL-5) along with the integration limits and expected accuracy of the result. The program then quickly computes the integral.

MU-17 LISTING FOR f(x)

RST

LRN

ENTER f(x):

LOC	CODE	KEY
000	42	STO
001	00	00
002	53	(
003	43	RCL
004	00	00
005	75	-
006	01	1
007	54	)
008	42	STO
009	06	06
010	91	R/S
011	76	LBL
012	16	A'
013	42	STO
014	01	01
015	53	(
016	43	RCL
017	01	01
018	45	Y^X
019	43	RCL
020	06	06
021	54	)
022	42	STO
023	02	02
024	53	(
025	53	(
026	43	RCL
027	01	01
028	94	+/-
029	22	INV
030	23	LNX
031	54	)
032	65	×
033	43	RCL
034	02	02
035	54	)
036	92	RTN

LRN

RST

ENTER	PRESS	DISPLAY
" α "	R/S	"α-1"
2nd Pgm 17		
a, lower limit	A	
b, upper limit	B	
ε	C	
	D	Integral

NOTE: $b \leq 227$; otherwise underflow will occur in computing f(x)

EXAMPLES:

FIND: γ (5.64, 8)

ε	γ (5.64, 8)	APPROXIMATE EXECUTION TIME
.01	55.57467363	45 sec
1E-3*	55.574674	1 min
1E-4	55.572748	1 min 20 sec
1E-5	55.57279	2 min 30 sec
1E-6	55.57279	4 min

FIND: Q (5.64, 8)

b = upper limit

ε	b	Q (5.64, 8)	APPROPRIATE EXECUTION TIME
1E-5	25	10.140547	2 min 30 sec
1E-6	50	10.140599	5 min
1E-6	200	10.140599	30 min

* 1E-3 = 10^{-3} ; 1E-4 = 10^{-4} ; etc.

NOTE: $\Gamma(5.64) = 65.71338 = 55.57279 + 10.140599$

PL-6

BASIC LANGUAGE PROGRAM LISTING FOR COMPUTING THE INCOMPLETE GAMMA FUNCTIONS USING SIMPSON'S NUMERICAL INTEGRATION TECHNIQUE

$\gamma(\alpha,\tau)$; $Q(\alpha,\tau)$

```
10 REM A = ALPHA
20 REM T = TAU
30 REM I = INCOMPLETE GAMMA FUNCTION WITH ARGUMENT ALPHA, TAU
40 REM XØ = LOWER INTEGRAL LIMIT
50 REM X1 = UPPER INTEGRAL LIMIT
60 REM N = NUMBER OF INTEGRATION SUBINTERVALS(MUST BE EVEN NUMBER)
70 REM INTEGRATION CALCULATIONS ARE BASED ON SIMPSON'S RULE
80 A=5.64
90 T=8.Ø
100 XØ=Ø
110 X1=8
120 N=32
130 H=(X1-XØ)/N
140 S1=Ø
150 S2=Ø
160 DIM Y(5ØØ)
170 X=XØ
180 Y(Ø)=(X↑(A-1))*(EXP(-X))
190 N2=N-1
200 Y(N)=(X1↑(A-1))*(EXP(-X1))
210 FOR J=1 TO N2
220 X=X+H
230 Y(J)=(X↑(A-1))*(EXP(-X))
240 NEXT J
250 FOR J=2 TO (N-2) STEP 2
260 S1=S1+Y(J)
270 NEXT J
280 S1=Y(Ø)+Y(N)+(2*S1)
290 FOR J=1 TO N2 STEP 2
300 S2=S2+Y(J)
310 NEXT J
320 S2=4*S2
330 I=(H/3)*(S1+S2)
340 PRINT "ALPHA =  ";A
350 PRINT "TAU   =  ";T
360 PRINT "X(LOWER) =   ";XØ
370 PRINT "X(UPPER) =   ";X1
380 PRINT "N =   ";N
390 PRINT
400 PRINT
410 PRINT "INTEGRAL =   ";I
420 END
```

*

HP-34C METHOD

The program for computing the Incomplete Gamma Functions on the HP-34C is listed in PL-15 which also computes other functions. The following listing for $\gamma(\alpha,\tau)$ and $Q(\alpha,\tau)$ are included for information only. This computer has a very fast and accurate numerical integration scheme built into it.

HP34C

INCOMPLETE GAMMA FUNCTIONS

$\gamma(\alpha,\tau)$, $Q(\alpha,\tau)$

$\gamma(\alpha,\tau)$:

Put value of α in X-register

Press ENTER ↑

Put value of τ in X-register *

Press GTO 4

Press R/S

Display will be $\gamma(\alpha,\tau)$

$Q(\alpha,\tau)$:

Put value of α in X-register

Press ENTER ↑

Put value of τ in X-register *

Press GTO 3

Press R/S

Display will be $Q(\alpha,\tau)$

<u>NOTE</u>: The upper integration limit that represents ∞ is stored in $R_{.1}$. For example: upper limit = 50. Put 50 in the X-register. Press STO .1 . The upper limit should be less than 227; otherwise underflow will occur in computing f(x). An upper limit of 50 has proven to give accurate results. See page 44.

* Press f SCI 7 (for accuracy of 7 significant figures)

EXAMPLES

FIND: $\gamma(5.64, 8)$

ACCURACY	$\gamma(5.64, 8)$	APPROXIMATE EXECUTION TIME
f SCI 5	55.5728	4 min
f SCI 7	55.57279	7 min

FIND: Q(5.64, 8)

Upper limit (i.e. ∞) = 50

ACCURACY	Q(5.64, 8)	APPROXIMATE EXECUTION TIME
f SCI 5	10.1406	5 min
f SCI 6	10.14060	6 min
f SCI 7	10.14059	7 min

HP34C

INCOMPLETE GAMMA FUNCTIONS

```
LBL 4
STO 8
R↓
STO 9        γ(α,τ)
0
ENTER ↑
RCL 8
∫2
RTN

LBL 3
STO 8
R↓
STO 9
RCL 8        Q(α,τ)
ENTER ↑
RCL .1
∫2
RTN

LBL 2
STO .0
CHS
e^x
RCL .0
RCL 9        Common
1
-
y^x
X
RTN
```

HP-67 Method

Simpson's Numerical Integration Technique is most conveniently found as program 19-01 of Hewlett-Packard's Math Pac 1 package.

Included also in PL-7 is a program listing for computing the Incomplete Gamma Functions on the HP-67.

User Instructions

PL-7

INCOMPLETE GAMMA FUNCTIONS

1 α↑, τ, A → γ(α,τ) α↑, τ, E → Q(α,τ) 2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
1	Enter α into X Register	α			α
2	Enter ↑		4	1	α
3	Enter τ into X Register	τ			τ
	PRESS A		1	1	γ(α,τ)
	OR				
1	Enter α into X Register	α			α
2	Enter ↑		4	1	α
3	Enter τ into X Register	τ			τ
4	PRESS E				Q(α,τ)

STEP	KEY ENTRY	KEY CODE	COMMENTS
001	LBL A	31 25 11	
	CL REG	31 43	
	STO 1	33 01	
	RV	35 53	
	STO 6	33 06	
	RCL 1	34 01	
	7	07	
	0	00	
	STO 9	33 09	
010	÷	81	
	STO 3	33 03	
	RCL 2	34 02	
	GSB C	31 22 13	
	STO +4	33 61 04	
	LBL B	31 25 12	
	RCL 3	34 03	
	STO +2	33 61 02	
	RCL 2	34 02	
	GSB C	31 22 13	
020	4	04	
	X	71	
	STO +4	33 61 04	
	2	02	
	STO +5	33 61 05	
	RCL 5	34 05	
	RCL 9	34 09	
	x = y	32 51	
	GTO D	22 14	
	RCL 3	34 03	
030	STO +2	33 61 02	
	RCL 2	34 02	
	GSB C	31 22 13	
	2	02	
	X	71	
	STO +4	33 61 04	
	GTO B	22 12	
	LBL D	31 25 14	
	RCL 3	34 03	
	STO +2	33 61 02	
040	RCL 2	34 02	
	GSB C	31 22 13	
	STO +4	33 61 04	
	RCL 4	34 04	
	3	03	
	÷	81	
	RCL 3	34 03	
	X	71	
	RTN	35 22	
	LBL C	31 25 13	
050	STO 7	33 07	
	RCL 6	34 06	
	1	01	
	–	51	
	y^x	35 63	
	RCL 7	34 07	
	CHS	42	
	e^x	32 52	
	X	71	
	RTN	35 22	
060	LBL E	31 25 15	
	CL REG	31 43	
	STO 1	33 01	
	V	35 53	
	STO 6	33 06	
	5	05	
	0	00	
	RCL 1	34 01	
	–	51	
	9	09	
070	8	08	
	STO 9	33 09	
	÷	81	
	STO 3	33 03	
	RCL 1	34 01	
	STO 2	33 02	
	GSB C	31 22 13	
	STO +4	33 61 04	
	GTO B	22 12	
080			
090			
100			
110			

REGISTERS

0	1	2	3	4	5	6	7	8	9
	USED	USED	USED	USED	USED	USED	USED		USED

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I

Program Description

Program Title Incomplete Gamma Functions By HP-67

Name **Date**

Address

City State New York Zip Code

Program Description, Equations, Variables, etc.

Program to evaluate the following two functions:

$$\gamma(\alpha,\tau) = \int_0^{\tau} x^{\alpha - 1} e^{-x} dx$$

$$Q(\alpha,\tau) = \int_{\tau}^{\infty} x^{\alpha - 1} e^{-x} dx$$

Comments: Running time for γ < 3 min, Q < 4 min

Operating Limits and Warnings Generally accurate to at least three significant figures. To increase accuracy for γ increase number in program lines 7 and 8 to a greater even number. To increase accuracy for calculating Q increase number on lines 65 and 66 (upper limit) and increase number on lines 69 and 70 to a greater even number.

DO NOT USE THIS SPACE

C. GAMMA FUNCTIONS COMPUTATION PROGRAM (BASIC LANGUAGE)

Included in PL-8 is a computer program written in Basic Language which quickly, conveniently and accurately computes the three Gamma Functions: $\Gamma(\alpha)$, $\gamma(a,t)$, $Q(\alpha,\tau)$.

The method for computing $\Gamma(\alpha)$ is as previously described. The method for computing the two Incomplete Gamma Functions has not been previously described. This method uses an approximation formula as described in Abramowitz [1] to calculate a "Normalized Incomplete Gamma Function: $Q_o(\alpha,\tau)$. $Q_o(\alpha,\tau)$ is extremely accurate for α = integer. τ may be either an integer or non-integer.

$$Q_o(\alpha,\tau) = \frac{Q(\alpha,\tau)}{(\alpha-1)!}$$

$$(\alpha-1)! = \Gamma(\alpha) \text{ for } \alpha = \text{integer}$$

The approximation formula is as follows:

$$Q_o(\alpha,\tau) = \frac{\tau^{\alpha-1} e^{-\tau}}{(\alpha-1)!}\left[1 + \sum_{\nu=2}^{\infty} \frac{(-1)^{\nu+1}}{\tau^{\nu-1}} \prod_{K=1}^{(\nu-1)} (K-\alpha)\right]$$

$$1 \leq \alpha \leq 350$$

NOTE: For most fatigue analysis applications $5 < \alpha < 15$.

C. GAMMA FUNCTIONS COMPUTATION PROGRAM (Cont'd)

For α = non-integer the interpolation method in Abramowitz [1] is used.

Define:

α_o = integer part of α

$w = \alpha - \alpha_o$

$$Q_o(\alpha,\tau) = Q_o(\alpha_o - 1, \tau)\left[\frac{1}{2}w^2 - \frac{1}{2}w\right]$$

$$+Q_o(\alpha_o, \tau)\left[1 - w^2\right]$$

$$+Q_o(\alpha_o + 1, \tau)\left[\frac{1}{2}w^2 + \frac{1}{2}w\right]$$

Then

$$Q(\alpha,\tau) = \Gamma(\alpha)\, Q_o(\alpha,\tau)$$

$$\gamma(\alpha,\tau) = \Gamma(\alpha)\left[1 - Q(\alpha,\tau)\right]$$

Several computation runs are included. It can be seen that the accuracy is sufficient for practical applications. Refer to Tables IV and V.

In PL-8

N = α

Z = τ

GØ = $Q_o(\alpha,\tau)$

G1 = $\gamma(\alpha,\tau)$

G2 = $Q(\alpha,\tau)$

G5 = $\Gamma(\alpha)$

PL-8

GAMMA FUNCTIONS COMPUTATION PROGRAM
(BASIC LANGUAGE)

```
10 REM Q0 IS THE NORMALIZED INCOMPLETE GAMMA FUNCTION
20 REM WITH INTEGRATION LIMITS TAU TO INFINITY AND
30 REM WITH ARGUMENT CHI-SQUARED,NU
40 REM
50 REM ALPHA MUST BE >= UNITY.
60 REM ALPHA= N
70 REM TAU= Z
80 REM
90 DEF FNG(N,Z)
100 IF N>1 THEN 130
110 Q0=EXP(-Z)
120 GO TO 480
130 H=Z/6
140 A=0
150 M=N
160 L=INT((N-4)/6)
170 FOR I=2 TO M
180 Y=G=1
190 FOR K=1 TO I-1
200 Y=(K-N)/Z*Y
210 V1=ABS(Y)
220 IF V1<1E-30 THEN 240
230 NEXT K
240 A=A+Y*(-1)^(I+1)
250 NEXT I
260 Q=A+1
270 IF N<10 THEN 430
280 J=0
290 FOR B=1 TO N-1
300 IF G<1E34 THEN 330
310 J=J+1
320 G=G/10
330 G=Z/B*G
340 IF B=L THEN 400
350 IF B=2*L THEN 400
360 IF B=3*L THEN 400
370 IF B=4*L THEN 400
380 IF B=5*L THEN 400
390 GO TO 410
400 G=G*EXP(-H)
410 NEXT B
420 GO TO 470
430 FOR B=1 TO N-1
440 G=Z/B*G
450 NEXT B
```

PL - 8

GAMMA FUNCTIONS COMPUTATION PROGRAM
(BASIC LANGUAGE)
CONTINUED

```
460 G=G*EXP(-H*5)
470 Q0=G*Q*10↑(J/2)*EXP(-H)*10↑(J/2)
480 FNG=Q0
490 FNEND
500 N=12.5
510 Z=12.5
520 W=N-INT(N)
530 IF W>0 THEN 560
540 G0=FNG(N,Z)
550 GO TO 640
560 L6=.5*(W↑2-W)
570 L7=1-W↑2
580 L8=.5*(W↑2+W)
590 N4=INT(N)-1
600 N5=INT(N)
610 N6=INT(N)+1
620 G0=FNG(N4,Z)*L6+FNG(N5,Z)*L7+FNG(N6,Z)*L8
630 GO TO 640
640 PRINT "ALPHA=";N
650 PRINT "TAU=";Z
660 PRINT
670 PRINT "NORM'D GAMMA FNC=";G0
680 PRINT
690 G4=1-G0
700 R=N
710 X=N
720 R=R+1
730 X=X*R
740 D=R-9
750 IF D>=0 THEN 770
760 GO TO 720
770 S=.9189385332
780 S=S+(R+.5)*LOG(R)-R
790 V=1-(1/(30*R↑2))+(1/(105*R↑4))
800 V=(1/(12*R))*V
810 S=S+V
820 G5=EXP(S)/X
830 PRINT "GAMMA(ALPHA)=";G5
840 PRINT
850 G1=G5*G4
860 PRINT "FIRST GAMMA FNC=";G1
870 G2=G5*G0
880 PRINT "SECOND GAMMA FNC=";G2
890 END
```

TABLE IV
COMPUTATION EXAMPLES
(α = non-integer)

```
ALPHA= 14.5
TAU=   18

NORM'D GAMMA FNC=   .1735187 (ACTUAL  .17356)

GAMMA(ALPHA)= 2.30923e 10

FIRST GAMMA FNC= 1.90854e 10
SECOND GAMMA FNC= 4.00695e 09


ready
*500 N=5.64
*510 Z=8
*RUN


ALPHA= 5.64
TAU=    8

NORM'D GAMMA FNC=   .1543015 (ACTUAL  .15429)

GAMMA(ALPHA)=   65.71338

FIRST GAMMA FNC=   55.57371
SECOND GAMMA FNC=   10.13967



ALPHA=  2.5
TAU=    2

NORM'D GAMMA FNC=   .5413411 (ACTUAL  .54924)

GAMMA(ALPHA)= 1.32934

FIRST GAMMA FNC=   .6097138
SECOND GAMMA FNC=   .7196266
```

TABLE V
COMPUTATION EXAMPLES
(α = integer)

ALPHA= 1
TAU= 1

NORM'D GAMMA FNC= .3678794 (ACTUAL .36788)

GAMMA(ALPHA)= .9999999

FIRST GAMMA FNC= .6321205
SECOND GAMMA FNC= .3678794

ALPHA= 2
TAU= 1.5

NORM'D GAMMA FNC= .5578254 (ACTUAL .55783)

GAMMA(ALPHA)= .9999999

FIRST GAMMA FNC= .4421746
SECOND GAMMA FNC= .5578254

ALPHA= 8
TAU= 7

NORM'D GAMMA FNC= .5987138 (ACTUAL .59871)

GAMMA(ALPHA)= 5040

FIRST GAMMA FNC= 2022.482
SECOND GAMMA FNC= 3017.517

ALPHA= 13
TAU= 17.5

NORM'D GAMMA FNC= .1116488 (ACTUAL .11165)

GAMMA(ALPHA)= 4.79001e 08

FIRST GAMMA FNC= 4.25521e 08
SECOND GAMMA FNC= 5.34800e 07

D. ERROR FUNCTION COMPUTATION

Definition

The Error Function used is that of Papoulis [2] :

$$\mathrm{erf}_p(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{\alpha} e^{-x^2/2}\, dx$$

$$\mathrm{erf}_p(0) = 0 \quad ; \quad \mathrm{erf}_p(\infty) = 0.5$$

$$\mathrm{erf}_p(-\alpha) = -\,\mathrm{erf}_p(\alpha)$$

Hereafter the above subscript p will be dropped for convenience. It will be understood that $\mathrm{erf}(\alpha) = \mathrm{erf}_p(\alpha)$.

Figure 5 shows a graph of $\mathrm{erf}(\alpha)$ versus α.

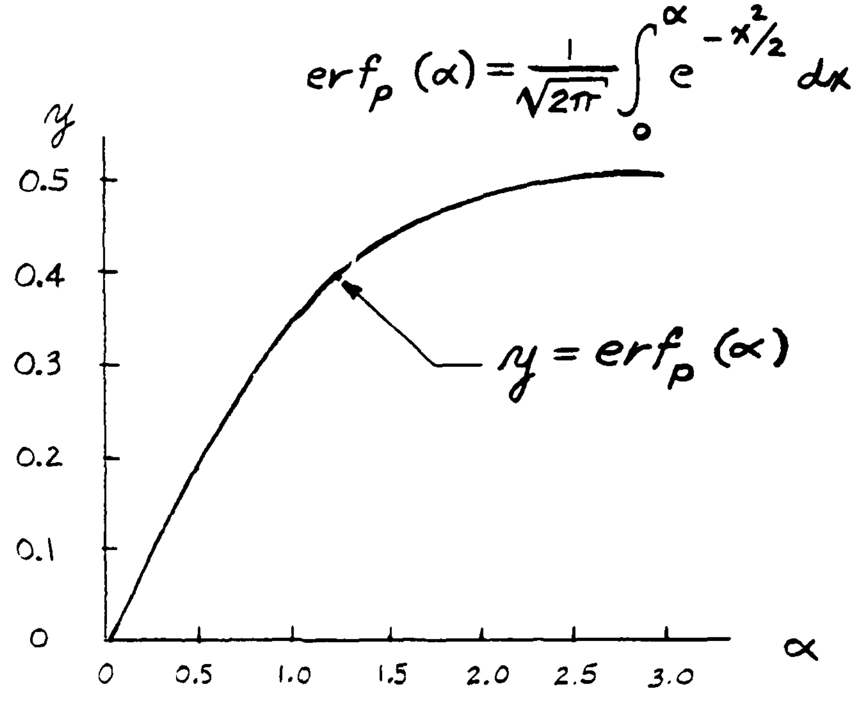


FIGURE 5 ERROR FUNCTION VERSUS α

TI-59 METHOD

The TI-59 Programmable Calculator can be used as the program listing PL-9 to calculate erf(α).

EXAMPLE:

FIND: erf (1.5)

SOLUTION: Use PL-9	Display
Enter 1.5	1.5
Press [A]	0.4331927713

erf(1.5) = 0.4331927713

EXAMPLE:

Find: erf(-0.125)

	Display
Enter .125	.125
Press [+/-]	-.125
Press [A]	-.0497382658

erf(-0.125) = -0.0497382658

ERROR FUNCTION COMPUTATION PROGRAM
(*TI-59*)

LOC	CODE	KEY
000	76	LBL
001	11	A
002	22	INV
003	86	STF
004	01	01
005	29	CP
006	77	GE
007	87	IFF
008	94	+/-
009	86	STF
010	01	01
011	76	LBL
012	87	IFF
013	53	(
014	42	STO
015	03	03
016	33	X^2
017	22	INV
018	23	LNX
019	65	×
020	02	2
021	65	×
022	89	π
023	54	)
024	34	√X
025	35	1/X
026	42	STO
027	01	01
028	93	.
029	02	2
030	03	3
031	01	1
032	06	6
033	04	4
034	01	1
035	09	9
036	49	PRD
037	03	03
038	01	1
039	44	SUM
040	03	03
041	43	RCL

LOC	CODE	KEY
042	03	03
043	35	1/X
044	53	(
045	53	(
046	42	STO
047	02	02
048	45	Y^X
049	04	4
050	65	×
051	01	1
052	93	.
053	03	3
054	03	3
055	00	0
056	02	2
057	07	7
058	04	4
059	04	4
060	02	2
061	09	9
062	75	-
063	43	RCL
064	02	02
065	45	Y^X
066	03	3
067	65	×
068	01	1
069	93	.
070	08	8
071	02	2
072	01	1
073	02	2
074	05	5
075	05	5
076	09	9
077	07	7
078	08	8
079	85	+
080	43	RCL
081	02	02
082	45	Y^X
083	02	2

ERROR FUNCTION COMPUTATION PROGRAM
(TI-59)

LOC	CODE	KEY
084	65	×
085	01	1
086	93	.
087	07	7
088	08	8
089	01	1
090	04	4
091	07	7
092	07	7
093	09	9
094	03	3
095	07	7
096	75	-
097	43	RCL
098	02	02
099	65	×
100	93	.
101	03	3
102	05	5
103	06	6
104	05	5
105	06	6
106	03	3
107	07	7
108	08	8
109	02	2
110	85	+
111	93	.
112	03	3
113	01	1
114	09	9
115	03	3
116	08	8
117	01	1
118	05	5
119	03	3
120	54	)
121	65	×
122	43	RCL
123	02	02
124	65	×
125	43	RCL

LOC	CODE	KEY
126	01	01
127	54	)
128	42	STO
129	04	04
130	87	IFF
131	01	01
132	01	01
133	41	41
134	53	(
135	94	+/-
136	85	+
137	93	.
138	05	5
139	54	)
140	91	R/S
141	53	(
142	43	RCL
143	04	04
144	75	-
145	93	.
146	05	5
147	54	)
148	91	R/S

HP-34C METHOD

The program listing for erf (α) is given in PL-15 and will not be shown here. The program is used as follows to calculate erf (α):

ERF(α)

GIVEN: Positive or negative value of α

FIND: ERF(α)

Put the value of α into X-register (i.e. Display register)

Key in desired accuracy *

Press B

Final display will be the value of erf(α)

*Accuracy: In scientific notation set the display for the value of α to the desired number of significant digits desired for the final value of erf(α).

EXAMPLE:

GIVEN: $\alpha = 1.$

FIND: ERF(α)

Put "1" in X-register

Press f SCI 7 (for accuracy of 7 significant figures)

Press B

ERF(α) = 0.3413447 will be displayed

HP-67 METHOD

A. MATH PAC 1

An Error Function Program is most conveniently obtained as program 18-02 of Hewlett-Packard's Math Pac 1 package. The Error Function of program 18-02 is defined slightly different than Papoulis' Error Function. Program 18-02 can still be used to calculate $\mathrm{erf}_p\ (\alpha)$ as follows:

a) Use α divided by $\sqrt{2}$ instead of α as the argument value.

b) Divide the final answer by 2.

B. PL-10

Program listing PL-10 computes both erf (α) and its inverse.

TABULAR METHODS

Table VI tabulates values of erf (α) versus α for values of α from zero to 4.0. These values were computed using the BASIC LANGUAGE Program as listed in PL-11.

EXAMPLE:

FIND erf (2.05)

erf(2.05) = 0.4798178

FIND erf (-1.72)

erf(1.72) = 0.4572838

erf(-1.72) = -0.4572838

FIND erf (2.044)

erf(2.045) = 0.4795726

erf(2.040) = 0.4793249

Using linear interpolation:

erf(2.044) = 0.4795230

$\mathrm{erf}(2.044)_{\mathrm{actual}} = 0.4795232$

TABULAR METHOD (Cont'd)

FIND: erf (0.068)

$erf(0.065) = 0.0259131$

$erf(0.070) = 0.0279032$

Using linear interpolation

$erf(0.068) = 0.0271071$

$erf(0.068)_{actual} = 0.0271072$

User Instructions

PL-10

PAPOULIS ERROR FUNCTION AND INVERSE HP-67

1		2
α,A → erf(α)		erf(α) ↑, ε, D → α

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
1	To calculate erf(α)	α			α
2	PRESS A		1	1	erf(α)
3	To calculate α given erf(α)	erf(α)	4	1	erf(α)
4	Input maximum allowable error	ε			ε
5	PRESS D (Runs about 10 minutes)		1	4	α

STEP	KEY ENTRY	KEY CODE	COMMENTS
001	LBL A	31 25 11	
	2	02	
	√	31 54	
	÷	81	
	STO 1	33 01	
	ENTER	41	
	X	71	
	2	02	
	X	71	
010	STO 2	33 02	
	1	01	
	STO 3	33 03	
	RCL 1	34 01	
	LBL 1	31 25 01	
	RCL 2	34 02	
	RCL 3	34 03	
	2	02	
	+	61	
	STO 3	33 03	
020	÷	81	
	RCL 1	34 01	
	X	71	
	STO 1	33 01	
	+	61	
	x ≠ y	32 61	
	GTO 1	22 01	
	2	02	
	X	71	
	π	35 73	
030	√	31 54	
	RCL 2	34 02	
	2	02	
	÷	81	
	e^x	32 52	
	X	71	
	÷	81	
	2	02	
	÷	81	
	RTN	35 22	
040	LBL B	31 25 12	
	RCL 5	34 05	
	RCL 4	34 04	
	−	51	
	STO 9	33 09	
	LBL 0	31 25 00	
	RCL 6	34 06	
	RCL 9	34 09	
	x ≤ y	32 71	
	GTO 6	22 06	
050	RCL 4	34 04	
	STO 7	33 07	
	LBL 2	31 25 02	
	RCL 5	34 05	
	x > y	32 81	
	GTO 3	22 03	
	•	83	
	5	05	
	STO x 9	33 71 09	
	GTO 0	22 00	
060	LBL 3	31 25 03	
	RCL 7	34 07	
	RCL 9	34 09	
	+	61	
	GSB C	31 22 13	
	STO 5	33 05	
	RCL 7	34 07	
	GSB C	31 22 13	
	RCL 5	34 05	
	X	71	
070	x < 0	31 74	
	GTO 4	22 04	
	RCL 7	34 07	
	RCL 9	34 09	
	+	61	
	GTO 2	22 02	
	LBL 4	31 25 04	
	•	83	
	5	05	
	STO x 9	33 71 09	
080	RCL 9	34 09	
	RCL 7	34 07	
	+	61	
	GSB C	31 33 13	
	STO 5	33 05	
	RCL 7	34 07	
	GSB C	31 22 13	
	RCL 5	34 05	
	X	71	
	x < 0	31 71	
090	GTO 5	22 05	
	RCL 9	34 09	
	STO + 7	33 61 07	
	LBL 5	31 25 05	
	RCL 6	34 06	
	RCL 9	34 09	
	x > y	32 81	
	GTO 4	22 04	
	RCL 7	34 07	
	RTN	35 22	
100	LBL 6	31 25 06	
	RCL 4	34 04	
	RTN	35 22	
	LBL D	31 25 14	
	CL REG	31 43	
	STO 6	33 06	
	V	35 53	
	STO 8	33 08	
	π	35 73	
	2	02	
110	+	61	
	STO 5	33 05	
	CHS	42	

REGISTERS

0	1	2	3	4	5	6	7	8	9
	USED	USED	USED	USED	USED	USED	USED	USED	USED
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I

Program Listing

STEP	KEY ENTRY	KEY CODE	COMMENTS
	•	83	
	1	01	
	–	51	
	STO 4	33 04	
	GTO B	22 12	
	LBL C	31 25 13	
	GSB A	31 22 11	
20	RCL 8	34 08	
	–	51	
	RTN	35 22	
30			
40			
50			
60			

STEP	KEY ENTRY	KEY CODE	COMMENTS
170			
180			
190			
200			
210			
220			

LABELS				
A USED	B USED	C USED	D USED	E
a	b	c	d	e
0 USED	1 USED	2 USED	3 USED	4 USED
5 USED	6 USED	7	8	9

FLAGS
0
1
2
3

SET STATUS

FLAGS (ON / OFF)	TRIG	DISP
0 ☐ ☐	DEG ☐	FIX ☒
1 ☐ ☐	GRAD ☐	SCI ☐
2 ☐ ☐	RAD ☐	ENG ☐
3 ☐ ☐		n 7

Program Description

Program Title Papoulis Error Function and Its Inverse - HP-67

Name **Date**

Address

City State Zip Code

Program Description, Equations, Variables, etc.

Evaluates $erf(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{\alpha} e^{\frac{-x^2}{2}} dx$

As well as the inverse function.

Operating Limits and Warnings The computing time for the inverse error function is long. It is a function of the maximum error ε desired.

For an error <.0005 the running time is about 10 minutes.

Range of α for inverse error function $-5 \rightarrow +5$.

DO NOT USE THIS SPACE

PL-11

ERROR FUNCTION PROGRAM LISTING
(BASIC LANGUAGE)

```
10 DEF FNE(X)
20 P=.2316419
30 T=1/(1+P*X)
40 B1=.31938153
50 B2=-.356563782
60 B3=1.781477937
70 B4=-1.821255978
80 B5=1.330274429
90 H=.39894228
100 Z=H*EXP(-.5*X↑2)
110 S=B1*T+B2*T↑2+B3*T↑3
120 S=S+B4*T↑4+B5*T↑5
130 Q=Z*S
140 FNE=.5-Q
150 FNEND
160 PRINT " A","ERF(A)"
170 FOR X=.005 TO 1 STEP .005
180 F=FNE(X)
190 PRINT X,F
200 NEXT X
210 END
```

$X = \alpha$
$F = erf(\alpha)$

TABLE VI ERROR FUNCTION VERSUS α

α	ERF(α)	α	ERF (α)
.005	.0019947	.235	.0928956
.01	.0039894	.24	.0948348
.015	.005984	.245	.0967717
.02	.0079784	.25	.0987063
.025	.0099726	.255	.1006384
.03	.0119665	.26	.102568
.035	.0139602	.265	.1044952
.04	.0159535	.27	.1064198
.045	.0179464	.275	.1083418
.05	.0197389	.28	.1102612
.055	.0219308	.285	.1121779
.06	.0239222	.29	.1140918
.065	.0259131	.295	.116003
.07	.0279032	.3	.1179113
.075	.0298927	.305	.1198168
.08	.0318814	.31	.1217174
.085	.0338694	.315	.1236191
.09	.0358565	.32	.1255158
.095	.0378427	.325	.1274094
.1	.0398279	.33	.1292999
.105	.0418122	.335	.1311874
.11	.0437954	.34	.1330717
.115	.0457775	.345	.1349528
.12	.0477585	.35	.1368306
.125	.0497383	.355	.1387051
.13	.0517168	.36	.1405764
.135	.0536941	.365	.1424442
.14	.05567	.37	.1443087
.145	.0576446	.375	.1461697
.15	.0596177	.38	.1480272
.155	.0615893	.385	.1498812
.16	.0635595	.39	.1517317
.165	.065528	.395	.1535785
.17	.067495	.4	.1554217
.175	.0694602	.405	.1572612
.18	.0714237	.41	.159097
.185	.0733855	.415	.160929
.19	.0753454	.42	.1627572
.195	.0773035	.425	.1645816
.2	.0792597	.43	.1664021
.205	.0812139	.435	.1682188
.21	.0831661	.44	.1700314
.215	.0851163	.445	.1718401
.22	.0870644	.45	.1736447
.225	.0890103	.455	.1754454
.23	.0909541	.46	.1772419

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)	α	ERF (α)
.465	.1790343	.69	.254903
.47	.1808225	.695	.2564724
.475	.1826065	.7	.2580364
.48	.1843863	.705	.259595
.485	.1861618	.71	.261148
.49	.187933	.715	.2626955
.495	.1896999	.72	.2642376
.5	.1914624	.725	.2657741
.505	.1932206	.73	.267305
.51	.1949743	.735	.2688303
.515	.1967235	.74	.2703501
.52	.1984682	.745	.2718642
.525	.2002084	.75	.2733727
.53	.2019441	.755	.2748756
.535	.2036751	.76	.2763728
.54	.2054015	.765	.2778643
.545	.2071233	.77	.2793501
.55	.2088403	.775	.2808302
.555	.2105527	.78	.2823046
.56	.2122603	.785	.2837733
.565	.2139631	.79	.2852362
.57	.2156612	.795	.2866933
.575	.2173544	.8	.2881447
.58	.2190427	.805	.2895902
.585	.2207262	.81	.29103
.59	.2224047	.815	.2924639
.595	.2240783	.82	.293892
.6	.2257469	.825	.295[illegible]142
.605	.2274106	.83	.2967307
.61	.2290691	.835	.2981412
.615	.2307227	.84	.2995459
.62	.2323712	.845	.3009446
.625	.2340145	.85	.3023375
.63	.2356528	.855	.3037245
.635	.2372859	.86	.3051055
.64	.2389138	.865	.3064806
.645	.2405365	.87	.3078498
.65	.242154	.875	.3092131
.655	.2437662	.88	.3105704
.66	.2453731	.885	.3119217
.665	.2469748	.89	.3132671
.67	.2485712	.895	.3146065
.675	.2501622	.9	.3159399
.68	.2517478	.905	.3172673
.685	.2533281		

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)	α	ERF (α)
.91	.3185888	1.135	.3718123
.915	.3199042	1.14	.3728568
.92	.3212136	1.145	.3738953
.925	.3225171	1.15	.374928
.93	.3238145	1.155	.3759547
.935	.3251059	1.16	.3769755
.94	.3263912	1.165	.3779904
.945	.3276706	1.17	.3789994
.95	.3289439	1.175	.3800026
.955	.3302112	1.18	.3809998
.96	.3314724	1.185	.3819712
.965	.3327276	1.19	.3829767
.97	.3339768	1.195	.3839564
.975	.3352197	1.2	.3849302
.98	.3364569	1.205	.3858983
.985	.337688	1.21	.3868605
.99	.3389129	1.215	.3878169
.995	.3401319	1.22	.3887675
1	.3413447	1.225	.3897123
1.005	.3425516	1.23	.3906513
1.01	.3437523	1.235	.3915846
1.015	.3449471	1.24	.3925122
1.02	.3461358	1.245	.393434
1.025	.3473184	1.25	.3943501
1.03	.348495	1.255	.3952605
1.035	.3496655	1.26	.3961652
1.04	.35083	1.265	.3970642
1.045	.3519885	1.27	.3979576
1.05	.3531409	1.275	.3988453
1.055	.3542873	1.28	.3997273
1.06	.3554277	1.285	.4006037
1.065	.356562	1.29	.4014746
1.07	.3576903	1.295	.4023398
1.075	.3588126	1.3	.4031994
1.08	.3599289	1.305	.4040535
1.085	.3610391	1.31	.404902
1.09	.3621434	1.315	.4057449
1.095	.3632416	1.32	.4065824
1.1	.3643339	1.325	.4074143
1.105	.3654201	1.33	.4082408
1.11	.3665004	1.335	.4090617
1.115	.3675747	1.34	.4098772
1.12	.368643	1.345	.4106873
1.125	.3697054	1.35	.4114919
1.13	.3707618	1.355	.4122911
		1.36	.4130849
		1.365	.4138734

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)	α	ERF (α)
1.37	.4146564	1.6	.4452007
1.375	.4154342	1.605	.4457531
1.38	.4162066	1.61	.4463011
1.385	.4169737	1.615	.4468446
1.39	.4177354	1.62	.4473839
1.395	.418472	1.625	.4479187
1.4	.4192432	1.63	.4484493
1.405	.4199893	1.635	.4489755
1.41	.4207301	1.64	.4494974
1.415	.4214656	1.645	.4500151
1.42	.4221961	1.65	.4505285
1.425	.4229213	1.655	.4510378
1.43	.4236414	1.66	.4515428
1.435	.4243563	1.665	.4520436
1.44	.4250662	1.67	.4525403
1.445	.425771	1.675	.4530329
1.45	.4264706	1.68	.4535214
1.455	.4271653	1.685	.4540057
1.46	.4278549	1.69	.454486
1.465	.4285374	1.695	.4549623
1.47	.429219	1.7	.4554345
1.475	.4298936	1.705	.4559028
1.48	.4305633	1.71	.4563671
1.485	.431228	1.715	.4568274
1.49	.4318878	1.72	.4572838
1.495	.4325427	1.725	.4577363
1.5	.4331927	1.73	.4581849
1.505	.4338379	1.735	.4586296
1.51	.4344783	1.74	.4590705
1.515	.4351138	1.745	.4595076
1.52	.4357445	1.75	.4599409
1.525	.4363704	1.755	.4603704
1.53	.4369916	1.76	.4607961
1.535	.4376081	1.765	.4612181
1.54	.4382198	1.77	.4616365
1.545	.4388268	1.775	.4620511
1.55	.4394292	1.78	.462462
1.555	.4400269	1.785	.4628694
1.56	.44062	1.79	.4632731
1.565	.4412085	1.795	.4636732
1.57	.4417924	1.8	.4640697
1.575	.4423718	1.805	.4644627
1.58	.4429466	1.81	.4648521
1.585	.4435168	1.815	.4652381
1.59	.4440826	1.82	.4656205
1.595	.4446439	1.825	.4659995
		1.83	.4663751

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)
1.835	.4667472
1.84	.4671159
1.845	.4674813
1.85	.4678433
1.855	.4682019
1.86	.4685573
1.865	.4689093
1.87	.4692581
1.875	.4696037
1.88	.469946
1.885	.4702851
1.89	.4706211
1.895	.4709538
1.9	.4712835
1.905	.47161
1.91	.4719334
1.915	.4722538
1.92	.4725711
1.925	.4728854
1.93	.4731966
1.935	.4735049
1.94	.4738102
1.945	.4741126
1.95	.474412
1.955	.4747085
1.96	.4750021
1.965	.4752929
1.97	.4755809
1.975	.475866
1.98	.4761483
1.985	.4764278
1.99	.4767046
1.995	.4769786
2	.4772499
2.005	.4775185
2.01	.4777845
2.015	.4780477
2.02	.4783084
2.025	.4785664
2.03	.4788218
2.035	.4790746
2.04	.4793249
2.045	.4795726
2.05	.4798178
2.055	.4800606
2.06	.4803008
2.065	.4805386
2.07	.4807739

α	ERF (α)
2.075	.4810068
2.08	.4812373
2.085	.4814654
2.09	.4816912
2.095	.4819146
2.1	.4821356
2.105	.4823544
2.11	.4825709
2.115	.4827851
2.12	.482997
2.125	.4832067
2.13	.4834142
2.135	.4836195
2.14	.4838227
2.145	.4840236
2.15	.4842224
2.155	.4844191
2.16	.4846137
2.165	.4848062
2.17	.4849966
2.175	.485185
2.18	.4853713
2.185	.4855556
2.19	.4857379
2.195	.4859182
2.2	.4860966
2.205	.486273
2.21	.4864475
2.215	.48662
2.22	.4867907
2.225	.4869594
2.23	.4871263
2.235	.4872914
2.24	.4874546
2.245	.487616
2.25	.4877756
2.255	.4879334
2.26	.4880894
2.265	.4882437
2.27	.4883962
2.275	.4885471
2.28	.4886962
2.285	.4888436
2.29	.4889894
2.295	.4891335
2.3	.4892759
2.305	.4894167
2.31	.4895559
2.315	.4896935

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)
2.32	.4898296
2.325	.489964
2.33	.4900969
2.335	.4902283
2.34	.4903581
2.345	.4904865
2.35	.4906133
2.355	.4907387
2.36	.4908625
2.365	.490985
2.37	.491106
2.375	.4912255
2.38	.4913437
2.385	.4914604
2.39	.4915758
2.395	.4916898
2.4	.4918025
2.405	.4919138
2.41	.4920237
2.415	.4921324
2.42	.4922397
2.425	.4923458
2.43	.4924506
2.435	.4925541
2.44	.4926564
2.445	.4927574
2.45	.4928572
2.455	.4929558
2.46	.4930531
2.465	.4931493
2.47	.4932443
2.475	.4933382
2.48	.4934309
2.485	.4935224
2.49	.4936128
2.495	.4937021
2.5	.4937903
2.505	.4938774
2.51	.4939634
2.515	.4940484
2.52	.4941322
2.525	.4942151
2.53	.4942969
2.535	.4943776
2.54	.4944574
2.545	.4945361
2.55	.4946138
2.555	.4946906
2.56	.4947664
2.565	.4948412
2.57	.494915
2.575	.494988
2.58	.49506
2.585	.495131
2.59	.4952012
2.595	.4952704
2.6	.4953388
2.605	.4954062
2.61	.4954728
2.615	.4955386
2.62	.4956035
2.625	.4956675
2.63	.4957307
2.635	.4957931
2.64	.4958547
2.645	.4959154
2.65	.4959754
2.655	.4960345
2.66	.4960929
2.665	.4961505
2.67	.4962074
2.675	.4962635
2.68	.4963188
2.685	.4963735
2.69	.4964273
2.695	.4964805
2.7	.496533
2.705	.4965847
2.71	.4966358
2.715	.4966862
2.72	.4967358
2.725	.4967849
2.73	.4968332
2.735	.4968809
2.74	.496928
2.745	.4969744
2.75	.4970202
2.755	.4970653
2.76	.4971099
2.765	.4971538
2.77	.4971971
2.775	.4972399
2.78	.497282
2.785	.4973236
2.79	.4973645
2.795	.497405
2.8	.4974448
2.805	.4974841

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)	α	ERF (α)
2.81	.4975229	3.055	.4988746
2.815	.4975611	3.06	.4988932
2.82	.4975988	3.065	.4989116
2.825	.4976359	3.07	.4989296
2.83	.4976725	3.075	.4989474
2.835	.4977086	3.08	.4989649
2.84	.4977443	3.085	.4989822
2.845	.4977794	3.09	.4989991
2.85	.497814	3.095	.4990159
2.855	.4978481	3.1	.4990323
2.86	.4978817	3.105	.4990485
2.865	.4979149	3.11	.4990645
2.87	.4979476	3.115	.4990802
2.875	.4979798	3.12	.4990957
2.88	.4980116	3.125	.4991109
2.885	.4980429	3.13	.4991259
2.89	.4980737	3.135	.4991407
2.895	.4981041	3.14	.4991552
2.9	.4981341	3.145	.4991695
2.905	.4981637	3.15	.4991836
2.91	.4981928	3.155	.4991974
2.915	.4982215	3.16	.4992111
2.92	.4982498	3.165	.4992245
2.925	.4982776	3.17	.4992377
2.93	.4983051	3.175	.4992508
2.935	.4983322	3.18	.4992636
2.94	.4983589	3.185	.4992762
2.945	.4983852	3.19	.4992886
2.95	.4984111	3.195	.4993008
2.955	.4984366	3.2	.4993128
2.96	.4984617	3.205	.4993246
2.965	.4984865	3.21	.4993363
2.97	.4985109	3.215	.4993477
2.975	.498535	3.22	.499359
2.98	.4985587	3.225	.4993701
2.985	.498582	3.23	.499381
2.99	.498605	3.235	.4993917
2.995	.4986277	3.24	.4994023
3	.49865	3.245	.4994127
3.005	.498672	3.25	.4994229
3.01	.4986937	3.255	.499433
3.015	.498715	3.26	.4994429
3.02	.4987361	3.265	.4994526
3.025	.4987568	3.27	.4994622
3.03	.4987772	3.275	.4994716
3.035	.4987972	3.28	.4994809
3.04	.498817	3.285	.49949
3.045	.4988365	3.29	.499499
3.05	.4988557	3.295	.4995078

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)	α	ERF (α)
3.3	.4995165	3.545	.4998Ø37
3.3Ø5	.4995251	3.55	.4998Ø73
3.31	.4995335	3.555	.499811
3.315	.4995417	3.56	.4998145
3.32	.4995499	3.565	.499818
3.325	.4995578	3.57	.4998215
3.33	.4995657	3.575	.4998248
3.335	.4995734	3.58	.4998282
3.34	.4995811	3.585	.4998314
3.345	.4995885	3.59	.4998346
3.35	.4995957	3.595	.4998378
3.355	.4996Ø31	3.6	.49984Ø9
3.36	.49961Ø2	3.6Ø5	.4998439
3.365	.4996172	3.61	.4998469
3.37	.4996241	3.615	.4998498
3.375	.49963Ø9	3.62	.4998527
3.38	.4996375	3.625	.4998555
3.385	.4996441	3.63	.4998583
3.39	.49965Ø5	3.635	.499861
3.395	.4996568	3.64	.4998636
3.4	.499663	3.645	.4998663
3.4Ø5	.4996691	3.65	.4998688
3.41	.4996751	3.655	.4998714
3.415	.499681	3.66	.4998739
3.42	.4996868	3.665	.4998763
3.425	.4996925	3.67	.4998787
3.43	.4996982	3.675	.499881
3.435	.4997Ø37	3.68	.4998833
3.44	.4997Ø91	3.685	.4998856
3.445	.4997144	3.69	.4998878
3.45	.4997197	3.695	.49989
3.455	.4997248	3.7	.4998922
3.46	.4997299	3.7Ø5	.4998943
3.465	.4997348	3.71	.4998963
3.47	.4997397	3.715	.4998984
3.475	.4997445	3.72	.4999ØØ4
3.48	.4997492	3.725	.4999Ø23
3.485	.4997539	3.73	.4999Ø42
3.49	.4997584	3.735	.4999Ø61
3.495	.4997629	3.74	.49990Ø8
3.5	.4997673	3.745	.4999Ø98
3.5Ø5	.4997717	3.75	.4999116
3.51	.4997759	3.755	.4999133
3.515	.49978Ø1	3.76	.499915
3.52	.4997842	3.765	.4999167
3.525	.4997882	3.77	.4999183
3.53	.4997922	3.775	.49992
3.535	.4997961	3.78	.4999216
3.54	.4997999	3.785	.4999231
		3.79	.4999246

TABLE VI ERROR FUNCTION VERSUS α

α	ERF (α)
3.795	.4999261
3.8	.4999276
3.805	.4999291
3.81	.4999305
3.815	.4999319
3.82	.4999333
3.825	.4999346
3.83	.4999359
3.835	.4999372
3.84	.4999385
3.845	.4999397
3.85	.4999409
3.855	.4999421
3.86	.4999433
3.865	.4999444
3.87	.4999456
3.875	.4999467
3.88	.4999477
3.885	.4999488
3.89	.4999499
3.895	.4999509
3.9	.4999519
3.905	.4999529
3.91	.4999538
3.915	.4999548
3.92	.4999557
3.925	.4999566
3.93	.4999575
3.935	.4999584
3.94	.4999592
3.945	.4999601
3.95	.4999609
3.955	.4999617
3.96	.4999625
3.965	.4999633
3.97	.499964
3.975	.4999648
3.98	.4999655
3.985	.4999662
3.99	.4999669
3.995	.4999676
4	.4999683

E. INVERSE ERROR FUNCTION COMPUTATION

Definition

The Inverse Error Function is that value of α that yields a specified value of erf (α). That is, given the value of erf (α), find α.

TI-59 METHOD:

The following computation method applies to positive values of α.

For negative values of α use the relationship

$$\text{erf}(-\alpha) = -\text{erf}(\alpha)$$

That is, calculate erf (α) and change the sign of erf (α) for negative α values.

Define

$$K = \text{erf}(\alpha) \quad ; \; 0 < K < 0.5$$

$$z = \sqrt{-2 \ln(2K)}$$

$$\alpha = g_0 + g_1 z + g_2 z^2 + \ldots + g_{10} z^{10}$$

Where:

$g_0 = 6.55864\text{x}10^{-4}$

$g_1 = -0.02069$

$g_2 = 0.737563$

$g_3 = -0.207071$

$g_4 = -2.06851\text{x}10^{-2}$

$g_5 = 0.03444$

$g_6 = -1.17213\text{x}10^{-2}$

$g_7 = 2.10941\text{x}10^{-3}$

$g_8 = -2.18541\text{x}10^{-4}$

$g_9 = 1.23163\text{x}10^{-5}$

$g_{10} = -2.93138\text{x}10^{-7}$

The following listing PL-12 is for the TI-59 Programmable Calculator. It utilizes the equations previously described. To use: enter the value of erf (α); then press [A]. The computed value of α will ultimately be displayed. Table VI can be used as a check on the reasonableness of the answer. Computation time is approximately 10 seconds.

$$0 < \text{erf}\ (\alpha) < 0.5$$

For negative values of erf (α), calculate α for the positive value of erf (α). Then change the sign of α.

Example:

Find α for erf (α) = 0.4995

Enter 0.4995; Press [A]

Display: 3.29049

Thus α = 3.29049

Example:

Find α for erf (α) = - 0.4975

Enter 0.4975 ; Press [A]

Display: 2.8066

Thus α = - 2.8066

PL-12

INVERSE ERROR FUNCTION; LISTING FOR TI-59 PROGRAMMABLE CALCULATOR

LOC	CODE	KEY
000	76	LBL
001	11	A
002	42	STO
003	01	01
004	53	(
005	53	(
006	43	RCL
007	01	01
008	65	×
009	02	2
010	94	+/-
011	85	+
012	01	1
013	54	)
014	23	LNX
015	65	×
016	02	2
017	54	)
018	94	+/-
019	34	ΓX
020	42	STO
021	02	02
022	06	6
023	93	.
024	05	5
025	05	5
026	08	8
027	06	6
028	04	4
029	52	EE
030	04	4
031	94	+/-
032	42	STO
033	03	03
034	53	(
035	02	2
036	93	.
037	00	0
038	06	6
039	09	9
040	52	EE
041	02	2
042	94	+/-
043	65	×
044	43	RCL
045	02	02
046	54	)
047	94	+/-
048	44	SUM
049	03	03
050	07	7
051	93	.
052	03	3
053	07	7
054	05	5
055	06	6
056	03	3
057	52	EE
058	01	1
059	94	+/-
060	65	×
061	53	(
062	43	RCL
063	02	02
064	54	)
065	45	YX
066	02	2
067	54	)
068	44	SUM
069	03	03
070	02	2
071	93	.
072	00	0
073	07	7
074	00	0
075	07	7
076	01	1
077	52	EE
078	01	1
079	94	+/-
080	65	×
081	53	(
082	43	RCL
083	02	02
084	54	)
085	45	YX
086	03	3
087	54	)

FL-12

INVERSE ERROR FUNCTION;
LISTING FOR TI-59 PROGRAMMABLE CALCULATOR

LOC	CODE	KEY
088	94	+/-
089	44	SUM
090	03	03
091	53	(
092	02	2
093	93	.
094	00	0
095	06	6
096	08	8
097	05	5
098	01	1
099	52	EE
100	02	2
101	94	+/-
102	65	×
103	53	(
104	43	RCL
105	02	02
106	54	)
107	45	Y^X
108	04	4
109	54	)
110	94	+/-
111	44	SUM
112	03	03
113	53	(
114	03	3
115	93	.
116	04	4
117	04	4
118	04	4
119	52	EE
120	02	2
121	94	+/-
122	65	×
123	53	(
124	43	RCL
125	02	02
126	54	)
127	45	Y^X
128	05	5
129	54	)
130	44	SUM
131	03	03
132	53	(
133	01	1
134	93	.
135	01	1
136	07	7
137	02	2
138	01	1
139	03	3
140	52	EE
141	02	2
142	94	+/-
143	65	×
144	53	(
145	43	RCL
146	02	02
147	54	)
148	45	Y^X
149	06	6
150	54	)
151	94	+/-
152	44	SUM
153	03	03
154	53	(
155	02	2
156	93	.
157	01	1
158	00	0
159	09	9
160	04	4
161	01	1
162	52	EE
163	03	3
164	94	+/-
165	65	×
166	53	(
167	43	RCL
168	02	02
169	54	)
170	45	Y^X
171	07	7
172	54	)
173	44	SUM
174	03	03
175	53	(
176	02	2

PL-12

INVERSE ERROR FUNCTION;
LISTING FOR TI-59 PROGRAMMABLE CALCULATOR

LOC	CODE	KEY
177	93	.
178	01	1
179	08	8
180	05	5
181	04	4
182	01	1
183	52	EE
184	04	4
185	94	+/-
186	65	×
187	53	(
188	43	RCL
189	02	02
190	54	)
191	45	Y^X
192	08	8
193	54	)
194	94	+/-
195	44	SUM
196	03	03
197	53	(
198	01	1
199	93	.
200	02	2
201	03	3
202	01	1
203	06	6
204	03	3
205	52	EE
206	05	5
207	94	+/-
208	65	×
209	53	(
210	43	RCL
211	02	02
212	54	)
213	45	Y^X
214	09	9
215	54	)
216	44	SUM
217	03	03
218	53	(
219	02	2
220	93	.

LOC	CODE	KEY
221	09	9
222	03	3
223	01	1
224	03	3
225	08	8
226	52	EE
227	07	7
228	94	+/-
229	65	×
230	53	(
231	43	RCL
232	02	02
233	54	)
234	45	Y^X
235	01	1
236	00	0
237	54	)
238	94	+/-
239	44	SUM
240	03	03
241	43	RCL
242	03	03
243	91	R/S
244	61	GTO
245	01	01
246	10	10
247	81	RST

INVERSE ERROR FUNCTION (HP-67 METHOD)

Refer to PL-10.

INVERSE ERROR FUNCTION (HP-34C METHOD)

The program listing for the Inverse Error Function and other functions for the HP-34C is given in PL-15. Included here is a brief listing for erf (α) and its inverse as well as an example in its use.

HP34C
ERF (α) AND INVERSE (α)

$erf(\alpha) \rightarrow R_6$

```
LBL 0
 x²
 2
 ÷
CHS
 eˣ
RTN

LBL 1
GSB B
RCL 6
 -
STO 7
RTN

LBL B
STO 7
 0
ENTER ↑
RCL 7
∫0
 2
 π
 x
√
1/x
 x
RTN
```

HP-34C

INVERSE ERROR FUNCTION

GIVEN: ERF(α) ; positive or negative value

FIND: α

Put the value of erf(α) into the X-register

Key in desired accuracy *

Store the value of erf(α) into register 6

Put initial estimate on bounds of α . Upper and lower bounds must bracket true value of α . Lower bound is put into X-register. Then press ENTER ↑ . Then put upper bound in X-register. Press f SOLVE 1.

Execution time is approximately as follows:

Accuracy	Time
f SCI 3	7 minutes
f SCI 5	14 minutes
f SCI 7	24 minutes

EXAMPLE:

GIVEN: ERF(α) = -.4331927

FIND: α

Put -.4331927 in X-register

* Press f SCI 7 (for accuracy of 7 significant figures)

Press STO 6

Press 2 CHS (i.e. -2)

Press ENTER ↑ — bounds on answer

Press .1 CHS (i.e. -.1)

Press f SOLVE 1

Final Display = -1.500

F. PROBABILITY OF FAILURE COMPUTATION

TI-59 METHOD

Many probability of failure F(α) expressions are of the form

$F(\alpha) = 0.5 + erf(\alpha)$

This expression can be conveniently computed using PL-13. To use enter the value of α and press [A]. The value for F(α) will ultimately be displayed.

EXAMPLE:

FIND: F(α) for α = -2.305, -1.365, -0.68, -0.365, -0.125, 0.125, 0.365, 0.68, 1.365, 2.305

α	F(α)
-2.305	0.0105832588
-1.365	0.0861265831
-0.68	0.24825221581
-0.365	0.3575557601
-0.125	0.4502617342
0.125	0.5497382658
0.365	0.6424442399
0.68	0.7517478419
1.365	0.9138734169
2.305	0.9894167412

PROBABILITY OF FAILURE COMPUTATION
(TI-59)

LOC	CODE	KEY
000	76	LBL
001	11	A
002	22	INV
003	86	STF
004	01	01
005	29	CP
006	77	GE
007	87	IFF
008	94	+/-
009	86	STF
010	01	01
011	76	LBL
012	87	IFF
013	53	(
014	42	STO
015	03	03
016	33	X^2
017	22	INV
018	23	LNX
019	65	×
020	02	2
021	65	×
022	89	π
023	54	)
024	34	$\sqrt{X}$
025	35	1/X
026	42	STO
027	01	01
028	93	.
029	02	2
030	03	3
031	01	1
032	06	6
033	04	4
034	01	1
035	09	9
036	49	PRD
037	03	03
038	01	1
039	44	SUM
040	03	03
041	43	RCL
042	03	03
043	35	1/X
044	53	(
045	53	(
046	42	STO
047	02	02
048	45	Y^X
049	04	4
050	65	×
051	01	1
052	93	.
053	03	3
054	03	3
055	00	0
056	02	2
057	07	7
058	04	4
059	04	4
060	02	2
061	09	9
062	75	-
063	43	RCL
064	02	02
065	45	Y^X
066	03	3
067	65	×
068	01	1
069	93	.
070	08	8
071	02	2
072	01	1
073	02	2
074	05	5
075	05	5
076	09	9
077	07	7
078	08	8
079	85	+
080	43	RCL
081	02	02
082	45	Y^X
083	02	2

PROBABILITY OF FAILURE COMPUTATION
(TI-59)

LOC	CODE	KEY
084	65	×
085	01	1
086	93	.
087	07	7
088	08	8
089	01	1
090	04	4
091	07	7
092	07	7
093	09	9
094	03	3
095	07	7
096	75	-
097	43	RCL
098	02	02
099	65	×
100	93	.
101	03	3
102	05	5
103	06	6
104	05	5
105	06	6
106	03	3
107	07	7
108	08	8
109	02	2
110	85	+
111	93	.
112	03	3
113	01	1
114	09	9
115	03	3
116	08	8
117	01	1
118	05	5
119	03	3
120	54	)
121	65	×
122	43	RCL
123	02	02
124	65	×
125	43	RCL

LOC	CODE	KEY
126	01	01
127	54	)
128	87	IFF
129	01	01
130	01	01
131	37	37
132	53	(
133	94	+/-
134	85	+
135	01	1
136	54	)
137	91	R/S

BASIC LANGUAGE METHOD

PL-14 is a modification of PL-11to calculate F(α).

$$F(\alpha) = 0.5 + erf(\alpha)$$

TABULAR METHOD

Table VII tabulates F(α) versus α.

EXAMPLE:

FIND: F(-2.305) using Table VII

$F(-2.3) = 0.0107241$

$F(-2.31) = 0.010444$

Using linear interpolation

$F(-2.305) = 0.010584$

$F(-2.305)_{ACTUAL} = 0.010583$

EXAMPLE:

FIND: F(.368) using Table VII

$F(.36) = 0.6405764$; $F(.37) = 0.6443087$

Using linear interpolation: $F(.368) = 0.6435622$

$F(.368)_{ACTUAL} = 0.6435633$

FAILURE PROBABILITY PROGRAM LISTING
(BASIC LANGUAGE)

```
10 REM A=ALPHA
20 DEF FNE(X)
30 P=.2316419
40 T=1/(1+P*X)
50 B1=.31938153
60 B2=-.356563782
70 B3=1.781477937
80 B4=-1.821255978
90 B5=1.330274429
100 H=.39894228
110 Z=H*EXP(-.5*X↑2)
120 S=B1*T+B2*T↑2+B3*T↑3
130 S=S+B4*T↑4+B5*T↑5
140 Q=Z*S
150 FNE=.5-Q
160 FNEND
170 FOR A=-2 TO -1 STEP .01
180 IF A>=0 GO TO 220
190 X=ABS(A)
200 F=.5-FNE(X)
210 GO TO 230
220 F=.5+FNE(A)
230 PRINT A,F
240 NEXT A
250 END
```

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)	α	F(α)
-4	3.16873e-05	-3.53	.0002078
-3.99	3.30545e-05	-3.52	.0002158
-3.98	3.44738e-05	-3.51	.0002241
-3.97	3.59528e-05	-3.5	.0002327
-3.96	3.74913e-05	-3.49	.0002416
-3.95	3.90932e-05	-3.48	.0002508
-3.94	4.07584e-05	-3.47	.0002603
-3.93	4.24944e-05	-3.46	.0002701
-3.92	4.42937e-05	-3.45	.0002803
-3.91	4.61675e-05	-3.44	.0002909
-3.9	4.81158e-05	-3.43	.0003018
-3.89	5.01424e-05	-3.42	.0003132
-3.88	5.22509e-05	-3.41	.0003249
-3.87	5.44414e-05	-3.4	.000337
-3.86	5.67175e-05	-3.39	.0003495
-3.85	5.90831e-05	-3.38	.0003625
-3.84	6.15418e-05	-3.37	.0003759
-3.83	6.40973e-05	-3.36	.0003898
-3.82	6.67498e-05	-3.35	.0004041
-3.81	6.95102e-05	-3.34	.0004189
-3.8	7.23749e-05	-3.33	.0004343
-3.79	7.53514e-05	-3.32	.0004501
-3.78	7.84397e-05	-3.31	.0004665
-3.77	8.16509e-05	-3.3	.0004835
-3.76	8.49850e-05	-3.29	.000501
-3.75	8.84458e-05	-3.28	.0005191
-3.74	9.20407e-05	-3.27	.0005378
-3.73	9.57698e-05	-3.26	.0005571
-3.72	9.96441e-05	-3.25	.0005771
-3.71	.0001037	-3.24	.0005977
-3.7	.0001078	-3.23	.000619
-3.69	.0001122	-3.22	.000641
-3.68	.0001167	-3.21	.0006637
-3.67	.0001213	-3.2	.0006872
-3.66	.0001261	-3.19	.0007114
-3.65	.0001312	-3.18	.0007364
-3.64	.0001364	-3.17	.0007623
-3.63	.0001417	-3.16	.0007889
-3.62	.0001473	-3.15	.0008164
-3.61	.0001531	-3.14	.0008448
-3.6	.0001591	-3.13	.0008741
-3.59	.0001654	-3.12	.0009043
-3.58	.0001718	-3.11	.0009355
-3.57	.0001785	-3.1	.0009677
-3.56	.0001855	-3.09	.0010009
-3.55	.0001927	-3.08	.0010351
-3.54	.0002001	-3.07	.0010704

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)	α	F(α)
-3.06	.0011068	-2.59	.0047988
-3.05	.0011443	-2.58	.00494
-3.04	.001183	-2.57	.005085
-3.03	.0012228	-2.56	.0052336
-3.02	.0012639	-2.55	.0053862
-3.01	.0013063	-2.54	.0055426
-3	.00135	-2.53	.0057031
-2.99	.001395	-2.52	.0058678
-2.98	.0014413	-2.51	.0060366
-2.97	.0014891	-2.5	.0062097
-2.96	.0015383	-2.49	.0063872
-2.95	.0015889	-2.48	.0065691
-2.94	.0016411	-2.47	.0067557
-2.93	.0016949	-2.46	.0069469
-2.92	.0017502	-2.45	.0071428
-2.91	.0018072	-2.44	.0073436
-2.9	.0018659	-2.43	.0075494
-2.89	.0019263	-2.42	.0077603
-2.88	.0019884	-2.41	.0079763
-2.87	.0020524	-2.4	.0081975
-2.86	.0021183	-2.39	.0084242
-2.85	.002186	-2.38	.0086563
-2.84	.0022557	-2.37	.008894
-2.83	.0023275	-2.36	.0091375
-2.82	.0024012	-2.35	.0093867
-2.81	.0024771	-2.34	.0096418
-2.8	.0025552	-2.33	.0099031
-2.79	.0026355	-2.32	.0101704
-2.78	.002718	-2.31	.010444
-2.77	.0028029	-2.3	.0107241
-2.76	.0028901	-2.29	.0110106
-2.75	.0029798	-2.28	.0113038
-2.74	.003072	-2.27	.0116038
-2.73	.0031668	-2.26	.0119106
-2.72	.0032641	-2.25	.0122244
-2.71	.0033642	-2.24	.0125454
-2.7	.003467	-2.23	.0128737
-2.69	.0035726	-2.22	.0132093
-2.68	.0036812	-2.21	.0135525
-2.67	.0037926	-2.2	.0139034
-2.66	.0039071	-2.19	.0142621
-2.65	.0040246	-2.18	.0146287
-2.64	.0041453	-2.17	.0150034
-2.63	.0042693	-2.16	.0153863
-2.62	.0043965	-2.15	.0157775
-2.61	.0045271	-2.14	.0161773
-2.6	.0046612	-2.13	.0165857

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)	α	F(α)
-2.12	.0170029	-1.65	.0494714
-2.11	.0174291	-1.64	.0505025
-2.1	.0178643	-1.63	.0515507
-2.09	.0183088	-1.62	.0526161
-2.08	.0187627	-1.61	.0536989
-2.07	.0192261	-1.6	.0547992
-2.06	.0196992	-1.59	.0559174
-2.05	.0201821	-1.58	.0570534
-2.04	.0206751	-1.57	.0582075
-2.03	.0211782	-1.56	.0593799
-2.02	.0216916	-1.55	.0605707
-2.01	.0222155	-1.54	.0617801
-2	.02275	-1.53	.0630083
-1.99	.0232954	-1.52	.0642554
-1.98	.0238517	-1.51	.0655217
-1.97	.0244191	-1.5	.0668072
-1.96	.0249978	-1.49	.0681121
-1.95	.025588	-1.48	.0694366
-1.94	.0261898	-1.47	.0707809
-1.93	.0268033	-1.46	.0721451
-1.92	.0274289	-1.45	.0735293
-1.91	.0280665	-1.44	.0749337
-1.9	.0287165	-1.43	.0763585
-1.89	.0293789	-1.42	.0778039
-1.88	.030054	-1.41	.0792699
-1.87	.0307418	-1.4	.0807567
-1.86	.0314427	-1.39	.0822645
-1.85	.0321567	-1.38	.0837934
-1.84	.0328841	-1.37	.0853435
-1.83	.0336249	-1.36	.086915
-1.82	.0343794	-1.35	.088508
-1.81	.0351478	-1.34	.0901227
-1.8	.0359302	-1.33	.0917592
-1.79	.0367269	-1.32	.0934176
-1.78	.0375379	-1.31	.095098
-1.77	.0383635	-1.3	.0968005
-1.76	.0392038	-1.29	.0985254
-1.75	.0400591	-1.28	.1002726
-1.74	.0409294	-1.27	.1020423
-1.73	.0418151	-1.26	.1038347
-1.72	.0427162	-1.25	.1056498
-1.71	.0436329	-1.24	.1074877
-1.7	.0445654	-1.23	.1093486
-1.69	.0455139	-1.22	.1112325
-1.68	.0464786	-1.21	.1131395
-1.67	.0474596	-1.2	.1150697
-1.66	.0484572	-1.19	.1170232

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)	α	F(α)
-1.18	.1190001	-.71	.238852
-1.17	.1210005	-.7	.2419636
-1.16	.1230244	-.69	.245097
-1.15	.1250719	-.68	.2482521
-1.14	.1271431	-.67	.2514288
-1.13	.1292381	-.66	.2546268
-1.12	.1313569	-.65	.257846
-1.11	.1334995	-.64	.2610862
-1.1	.135666	-.63	.2643472
-1.09	.1378565	-.62	.2676288
-1.08	.140071	-.61	.2709308
-1.07	.1423096	-.6	.2742531
-1.06	.1445722	-.59	.2775953
-1.05	.146859	-.58	.2809573
-1.04	.1491699	-.57	.2843388
-1.03	.1515049	-.56	.2877397
-1.02	.1538641	-.55	.2911596
-1.01	.1562476	-.54	.2945985
-1	.1586553	-.53	.2980559
-.99	.1610871	-.52	.3015318
-.98	.1635431	-.51	.3050257
-.97	.1660232	-.5	.3085375
-.96	.1685276	-.49	.3120669
-.95	.1710561	-.48	.3156137
-.94	.1736088	-.47	.3191775
-.93	.1761855	-.46	.3227581
-.92	.1787863	-.45	.3263552
-.91	.1814112	-.44	.3299686
-.9	.1840601	-.43	.3335979
-.89	.1867329	-.42	.3372428
-.88	.1894296	-.41	.340903
-.87	.1921501	-.4	.3445783
-.86	.1948945	-.39	.3482683
-.85	.1976625	-.38	.3519727
-.84	.2004541	-.37	.3556913
-.83	.2032693	-.36	.3594236
-.82	.206108	-.35	.3631694
-.81	.20897	-.34	.3669283
-.8	.2118553	-.33	.3707
-.79	.2147638	-.32	.3744842
-.78	.2176954	-.31	.3782805
-.77	.2206499	-.3	.3820886
-.76	.2236272	-.29	.3859082
-.75	.2266273	-.28	.3897388
-.74	.2296499	-.27	.3935802
-.73	.232695	-.26	.3974319
-.72	.2357624	-.25	.4012937

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)	α	F(α)
-.24	.4051652	.23	.5909541
-.23	.4090459	.24	.5948348
-.22	.4129356	.25	.5987063
-.21	.4168339	.26	.6025681
-.2	.4207403	.27	.6064198
-.19	.4246546	.28	.6102612
-.18	.4285763	.29	.6140918
-.17	.432505	.3	.6179114
-.16	.4364405	.31	.6217194
-.15	.4403823	.32	.6255158
-.14	.44433	.33	.6292999
-.13	.4482832	.34	.6330717
-.12	.4522415	.35	.6368306
-.11	.4562046	.36	.6405764
-.1	.4601721	.37	.6443087
-.09	.4641435	.38	.6480272
-.08	.4681185	.39	.6517317
-.07	.4720968	.4	.6554217
-.06	.4760778	.41	.659097
-.05	.4800611	.42	.6627572
-.04	.4840465	.43	.6664021
-.03	.4880335	.44	.6700314
-.02	.4920216	.45	.6736447
-.01	.4960106	.46	.6772419
0	.5	.47	.6808225
.01	.5039894	.48	.6843863
.02	.5079784	.49	.6879331
.03	.5119665	.5	.6914625
.04	.5159535	.51	.6949743
.05	.5199389	.52	.6984682
.06	.5239222	.53	.7019441
.07	.5279032	.54	.7054015
.08	.5318814	.55	.7088403
.09	.5358565	.56	.7122603
.1	.5398279	.57	.7156612
.11	.5437954	.58	.7190427
.12	.5477585	.59	.7224047
.13	.5517168	.6	.7257469
.14	.55567	.61	.7290691
.15	.5596177	.62	.7323712
.16	.5635595	.63	.7356528
.17	.567495	.64	.7389138
.18	.5714237	.65	.7421539
.19	.5753454	.66	.7453731
.2	.5792597	.67	.7485712
.21	.5831661	.68	.7517478
.22	.5870644	.69	.754903

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)
.7	.7580364
.71	.761148
.72	.7642376
.73	.767305
.74	.7703501
.75	.7733727
.76	.7763728
.77	.7793501
.78	.7823046
.79	.7852362
.8	.7881447
.81	.79103
.82	.793892
.83	.7967307
.84	.7995459
.85	.8023375
.86	.8051055
.87	.8078498
.88	.8105704
.89	.8132671
.9	.8159399
.91	.8185888
.92	.8212136
.93	.8238145
.94	.8263912
.95	.8289439
.96	.8314724
.97	.8339768
.98	.8364569
.99	.8389129
1	.8413447
1.01	.8437523
1.02	.8461358
1.03	.848495
1.04	.85083
1.05	.8531409
1.06	.8554277
1.07	.8576903
1.08	.8599289
1.09	.8621434
1.1	.8643339
1.11	.8665004
1.12	.868643
1.13	.8707618
1.14	.8728568
1.15	.874928
1.16	.8769755
1.17	.8789994
1.18	.8809998
1.19	.8829767
1.2	.8849302
1.21	.8868605
1.22	.8887675
1.23	.8906513
1.24	.8925122
1.25	.8943501
1.26	.8961652
1.27	.8979576
1.28	.8997273
1.29	.9014746
1.3	.9031994
1.31	.904902
1.32	.9065824
1.33	.9082408
1.34	.9098772
1.35	.9114919
1.36	.9130849
1.37	.9146564
1.38	.9162066
1.39	.9177354
1.4	.9192432
1.41	.92073
1.42	.9221961
1.43	.9236414
1.44	.9250662
1.45	.9264706
1.46	.9278549
1.47	.929219
1.48	.9305633
1.49	.9318878
1.5	.9331927
1.51	.9344782
1.52	.9357444
1.53	.9369916
1.54	.9382198
1.55	.9394292
1.56	.94062
1.57	.9417924
1.58	.9429466
1.59	.9440826
1.6	.9452007
1.61	.9463011
1.62	.9473839
1.63	.9484492

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)	α	F(α)
1.64	.9494974	2.11	.9825709
1.65	.9505285	2.12	.982997
1.66	.9515428	2.13	.9834142
1.67	.9525403	2.14	.9838227
1.68	.9535213	2.15	.9842224
1.69	.954486	2.16	.9846137
1.7	.9554346	2.17	.9849966
1.71	.9563671	2.18	.9853713
1.72	.9572838	2.19	.9857379
1.73	.9581849	2.2	.9860966
1.74	.9590705	2.21	.9864475
1.75	.9599409	2.22	.9867906
1.76	.9607961	2.23	.9871263
1.77	.9616365	2.24	.9874546
1.78	.962462	2.25	.9877756
1.79	.9632731	2.26	.9880894
1.8	.9640697	2.27	.9883962
1.81	.9648521	2.28	.9886962
1.82	.9656205	2.29	.9889894
1.83	.9663751	2.3	.9892759
1.84	.9671159	2.31	.9895559
1.85	.9678433	2.32	.9898296
1.86	.9685573	2.33	.9900969
1.87	.9692581	2.34	.9903581
1.88	.969946	2.35	.9906133
1.89	.9706211	2.36	.9908625
1.9	.9712835	2.37	.991106
1.91	.9719334	2.38	.9913437
1.92	.9725711	2.39	.9915758
1.93	.9731966	2.4	.9918025
1.94	.9738102	2.41	.9920237
1.95	.974412	2.42	.9922397
1.96	.9750021	2.43	.9924506
1.97	.9755809	2.44	.9926564
1.98	.9761483	2.45	.9928572
1.99	.9767046	2.46	.9930531
2	.9772499	2.47	.9932443
2.01	.9777845	2.48	.9934309
2.02	.9783084	2.49	.9936128
2.03	.9788218	2.5	.9937903
2.04	.9793249	2.51	.9939634
2.05	.9798178	2.52	.9941322
2.06	.9803008	2.53	.9942968
2.07	.9807739	2.54	.9944573
2.08	.9812373	2.55	.9946138
2.09	.9816912	2.56	.9947664
2.1	.9821356	2.57	.994915

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)	α	F(α)
2.58	.99506	3.05	.9988557
2.59	.9952012	3.06	.9988932
2.6	.9953388	3.07	.9989296
2.61	.9954728	3.08	.9989649
2.62	.9956035	3.09	.9989991
2.63	.9957307	3.1	.9990323
2.64	.9958547	3.11	.9990645
2.65	.9959754	3.12	.9990957
2.66	.9960929	3.13	.9991259
2.67	.9962074	3.14	.9991552
2.68	.9963188	3.15	.9991836
2.69	.9964273	3.16	.9992111
2.7	.996533	3.17	.9992377
2.71	.9966358	3.18	.9992636
2.72	.9967358	3.19	.9992886
2.73	.9968332	3.2	.9993128
2.74	.996928	3.21	.9993363
2.75	.9970202	3.22	.999359
2.76	.9971099	3.23	.999381
2.77	.9971971	3.24	.9994023
2.78	.997282	3.25	.9994229
2.79	.9973645	3.26	.9994429
2.8	.9974448	3.27	.9994622
2.81	.9975229	3.28	.9994809
2.82	.9975988	3.29	.999499
2.83	.9976725	3.3	.9995165
2.84	.9977443	3.31	.9995335
2.85	.997814	3.32	.9995499
2.86	.9978817	3.33	.9995657
2.87	.9979476	3.34	.999581
2.88	.9980116	3.35	.9995959
2.89	.9980737	3.36	.9996102
2.9	.9981341	3.37	.9996241
2.91	.9981928	3.38	.9996375
2.92	.9982498	3.39	.9996505
2.93	.9983051	3.4	.999663
2.94	.9983589	3.41	.9996751
2.95	.9984111	3.42	.9996868
2.96	.9984617	3.43	.9996982
2.97	.9985109	3.44	.9997091
2.98	.9985587	3.45	.9997197
2.99	.998605	3.46	.9997299
3	.99865	3.47	.9997397
3.01	.9986937	3.48	.9997492
3.02	.9987361	3.49	.9997584
3.03	.9987772	3.5	.9997673
3.04	.998817	3.51	.9997759

TABLE VII FAILURE PROBABILITY VERSUS α

α	F(α)
3.52	.9997842
3.53	.9997922
3.54	.9997999
3.55	.9998073
3.56	.9998145
3.57	.9998215
3.58	.9998282
3.59	.9998346
3.6	.9998408
3.61	.9998469
3.62	.9998527
3.63	.9998583
3.64	.9998636
3.65	.9998688
3.66	.9998739
3.67	.9998787
3.68	.9998833
3.69	.9998878
3.7	.9998922
3.71	.9998963
3.72	.9999004
3.73	.9999042
3.74	.999908
3.75	.9999116
3.76	.999915
3.77	.9999183
3.78	.9999216
3.79	.9999246
3.8	.9999276
3.81	.9999305
3.82	.9999333
3.83	.9999359
3.84	.9999385
3.85	.9999409
3.86	.9999433
3.87	.9999456
3.88	.9999477
3.89	.9999499
3.9	.9999519
3.91	.9999538
3.92	.9999557
3.93	.9999575
3.94	.9999592
3.95	.9999609
3.96	.9999625
3.97	.999964
3.98	.9999655

α	F(α)
3.99	.9999669
4	.9999683

G. HP-34C PROGRAMS

The following program listing PL-15 is to be used to compute the Incomplete Gamma Functions, erf (α) and its inverse, and to solve the accelerated test rranscendental function (which is discussed in a later section).

PL-15

HP-34C

```
25 13 11    LBL A        Transcendental
   23  5    STO 5        Function
   24  3    RCL 3        f(x)
   25  3    y^x
   24  0    RCL 0
      61    x
      32    CHS
       1    1
      51    +
   24  4    RCL 4
   25  3    y^x
   24  2    RCL 2
      61    x
      32    CHS
   24  5    RCL 5
      51    +
   25 12    RTN
25 13  0    LBL 0        erf(α) +
   15  3    x^2          erf^-1(α)
       2    2
      71    ÷
      32    CHS
   15  1    e^x
   25 12    RTN
25 13  1    LBL 1
   13 12    GSB B
   24  6    RCL 6
      41    -
   25 12    RTN
25 13 12    LBL B
   23  7    STO 7
       0    0
      31    ENTER ↑
   24  7    RCL 7
14 72  0    ∫0
       2    2
   25 73    π            erf(α) +
      61    x            erf^-1(α)
   14  3    √
   25  2    1/x
      61    x
   25 12    RTN
25 13  4    LBL 4        Incomplete Gamma Functions
   23  8    STO 8        γ(α,τ)
   15 22    R↓
   23  9    STO 9
       0    0
      31    ENTER ↑
   24  8    RCL 8
14 72  2    ∫2
   25 12    RTN
```

Key code	Key	
25 13 3	LBL 3	Q(α,τ)
23 8	STO 8	
15 22	R↓	
23 9	STO 9	
24 8	RCL 8	
31	ENTER ↑	
24 .1	RCL .1	
14 72 2	∫2	
25 12	RTN	
25 13 2	LBL 2	Common
23 .0	STO .0	
32	CHS	
15 1	e^x	
24 .0	RCL .0	
24 9	RCL 9	
1	1	
41	-	
25 3	y^x	
61	x	
25 12	RTN	

H. ACCELERATED RANDOM TEST LEVEL COMPUTATION

The computation methods in this section calculate the accelerated random input vibration level $\ddot{x}_2$ that will cumulate the same fatigue damage to a structural element being stressed for T2 hours as a random input vibration level $\ddot{x}_1$ for T1 hours. Fracture Mechanics effects causes the relationship between parameters to be a transcendental function as follows:

$$\ddot{x}_2 - b_4 \left[1 - b_2 \, \ddot{x}_2^{\,\eta(\theta-2)} \right]^{1/\eta\theta} = 0$$

$\ddot{x}_2$ is computed by solving the above function. These programs use the half-interval technique.

INPUT PARAMETERS:

- a_i = initial crack half-length (inches); NOTE: a_i must be >0.004 inches
- $\ddot{x}_1$ = input acceleration rms level at the service environment (g rms)
- $\ddot{x}_2$ = input acceleration rms level at the accelerated test environment (g rms)
- C_4 = rms stress per g rms constant (ksi/g rms)
- T_1 = service environment duration (time units)
- T_2 = accelerated test duration (same time units as T_1).
- η = damping linearity constant
- θ = material crack growth rate constant
- Y = geometrical parameter
- $\overline{A}$ = material sinusoidal fatigue curve constant (ksi)
- $\overline{C}$ = material random fatigue curve constant (ksi)
- ΔK_c = material fracture toughness (ksi $\sqrt{in}$)

The constant C4 is the ratio of the rms stress in the structural element that is cumulating fatigue damage to the "black box" rms vibration input acceleration level. These programs assume that the power spectral density of the input acceleration retains the same shape at both $\ddot{x}_1$ and $\ddot{x}_2$ levels, expecially in the vicinity of structural resonant frequencies. Otherwise C4 will take on a different value at each input acceleration level. This is because the rms stress level in the structural element (hence, fatigue) is related to the square root of the power spectral density of the acceleration input in the regions of the resonant frequencies.

The constant η denotes the linearity of the structural assembly in relating the rms stress level σ of the structural element to the input acceleration level as follows:

$$\sigma_1 = C4\ \ddot{x}_1^{\eta} \qquad ; \qquad \sigma_2 = C4\ \ddot{x}_2^{\eta}$$

$\eta = 1$ represents the damping linearity system. $\eta = 0.833$ represents a system whose damping is controlled by internal stress-strain hysteresis damping. $\eta = 1.5$ represents a system controlled by Coulomb friction damping.

Define

CONSTANTS:

$$b_1 = \frac{1}{a_i}\left[\frac{\Delta K_c}{C_5 Y}\right]^2$$

$$C_5 = \left(\frac{2\bar{A}}{\bar{C}}\right) C_4$$

$$b_2 = \left(\frac{1}{b_1}\right)^{\frac{\theta - 2}{2}}$$

$$b_3 = 1 - b_2\, \ddot{x}_1^{\;\eta(\theta - 2)}$$

$$b_4 = \ddot{x}_1 \left(\frac{T_1}{T_2}\right)^{1/\eta\theta} \left(\frac{1}{b_3}\right)^{1/\eta\theta}$$

$$\ddot{x}_{2_{max}} = b_1^{\;1/2\eta}$$

For convenience let $\ddot{x}_2$ be also referred to as x. The value of x (i.e. $\ddot{x}_2$) to be solved for is that value that sets the following function equal to zero:

$$f(x) = x - b_4 \left[1 - b_2\, x^{\eta(\theta - 2)} \right]^{1/\eta\theta}$$

$\ddot{x}_{2_{max}}$ is the largest value of x that will not cause computational problems (e.g. ℓn of a negative number). It represents the largest value of x that has practical use. If $\ddot{x}_2 > \ddot{x}_{2_{max}}$, fatigue failure will occur during the application of the first vibration cycle. A similar practical limit is imposed on the selection of the value for a_i. If a_i is chosen larger than a_{c_2} (see below), $\ddot{x}_2$ will be less than $\ddot{x}_1$, which is meaningless for an "accelerated" test.

a_{c_2} = critical crack size at $\ddot{x}_2$

$$a_{c_2} = \left[\frac{\Delta K_c}{Y\ C_5\ \ddot{x}_2^{\,\eta}}\right]^2 \quad \text{inches}$$

1. BASIC LANGUAGE (PL-16)

PL-16 solves for $\ddot{x}_2$.

INPUT DATA:

A1, C4, N, O, K, Y, C, A, T2, T1, X1

represents

a_i, C4, η, θ, ΔK_c, Y, $\overline{C}$, $\overline{A}$, T_2, T_1, $\ddot{x}_1$

OUTPUT DATA:

X2 = $\ddot{x}_2$

EXAMPLES:

```
RUN


WHAT ARE A1,C4,N,O,K,Y,C,A,T2,T1,X1
?.007,1,.833,4,20,1.77,80,180,1,1000,1
X2=    7.869127

ready
*RUN


WHAT ARE A1,C4,N,O,K,Y,C,A,T2,T1,X1
?.6,1,.833,4,20,1.77,80,180,1,1000,1
X2=    .893453
```

```
10 PRINT " WHAT ARE A1,C4,N,O,K,Y,C,A,T2,T1,X1"
20 INPUT A1,C4,N,O,K,Y,C,A,T2,T1,X1
30 E=1E-9
40 C5=2*A*C4/C
50 B1=(K/(C5*Y))**2/A1
60 B2=(1/B1)**((O-2)/2)
70 B3=X1**(N*(O-2))*(-B2)+1
80 B4= (1/B3)**(1/(N*O))
90 B4=B4*X1*(T1/T2)**(1/(N*O))
100 M=B1**(1/(2*N))
110 DEF FNX(X)=X-B4*(1-B2*X**(N*(O-2)))**(1/(N*O))
120 H=M
130 X4=FNX(X)*FNX(H)
140 IF X4>0 THEN 230
150 H=H/2
160 X3=FNX(X)*FNX(X+H)
170 IF X3< 0 THEN 190
180 X=X+H
190 IF H<E THEN 210
200 GO TO 150
210 PRINT "X2=";X
220 GO TO 240
230 PRINT"PHYSICALLY IMPOSSIBLE DATA INPUT SET"
240 END
```

2. TI-59 (PL-17)

Enter PL-17 into the computer; then execute the program as shown in the following example :

INPUT PARAMETER	ENTER	PRESS
a_i	7×10^{-3}	R/S
$\ddot{x}_1$	1	R/S
C_4	1	R/S
T_1/T_2	10^3	R/S
η	.833	R/S
θ	4	R/S
$\bar{A}$	180	R/S
$\bar{C}$	80	R/S
Y	1.77	R/S
ΔK_c	20	R/S

PRESS	DISPLAY
R/S	$\ddot{x}_2 = 7.8722755$

For $a_i = 0.6$ inches $\ddot{x}_2 = 3.8934211$ g rms

Execution time $\simeq$ 2 minutes

TI-59 PL-17 LISTING

LRN

LOC	CODE	KEY
000	42	STO
001	00	00
002	91	R/S
003	42	STO
004	01	01
005	91	R/S
006	42	STO
007	02	02
008	91	R/S
009	42	STO
010	03	03
011	91	R/S
012	42	STO
013	04	04
014	91	R/S
015	42	STO
016	05	05
017	91	R/S
018	42	STO
019	06	06
020	91	R/S
021	42	STO
022	07	07
023	91	R/S
024	42	STO
025	08	08
026	91	R/S
027	42	STO
028	11	11
029	91	R/S
030	53	(
031	02	2
032	65	×
033	43	RCL
034	04	04
035	54	)
036	35	1/X
037	42	STO
038	13	13
039	53	(
040	02	2
041	65	×
042	43	RCL
043	06	06
044	55	÷
045	43	RCL
046	07	07
047	54	)
048	42	STO
049	14	14
050	53	(
051	53	(
052	53	(
053	43	RCL
054	11	11
055	55	÷
056	43	RCL
057	14	14
058	55	÷
059	43	RCL
060	08	08
061	54	)
062	54	)
063	33	X²
064	54	)
065	55	÷
066	43	RCL
067	00	00
068	54	)
069	42	STO
070	16	16
071	53	(
072	02	2
073	65	×
074	43	RCL
075	04	04
076	54	)
077	42	STO
078	17	17
079	53	(
080	53	(
081	43	RCL
082	05	05
083	75	−
084	02	2
085	54	)
086	55	÷
087	02	2
088	54	)
089	42	STO
090	20	20
091	53	(
092	53	(
093	43	RCL
094	05	05
095	75	−
096	02	2
097	54	)
098	65	×
099	43	RCL

LOC	CODE	KEY	LOC	CODE	KEY
100	04	04	150	43	RCL
101	54	)	151	03	03
102	42	STO	152	45	YX
103	21	21	153	43	RCL
104	53	(	154	22	22
105	43	RCL	155	54	)
106	04	04	156	65	×
107	65	×	157	53	(
108	43	RCL	158	53	(
109	05	05	159	43	RCL
110	54	)	160	34	34
111	35	1/X	161	35	1/X
112	42	STO	162	54	)
113	22	22	163	45	YX
114	53	(	164	43	RCL
115	53	(	165	22	22
116	43	RCL	166	54	)
117	16	16	167	54	)
118	35	1/X	168	42	STO
119	54	)	169	25	25
120	45	YX	170	53	(
121	43	RCL	171	53	(
122	20	20	172	43	RCL
123	54	)	173	16	16
124	42	STO	174	45	YX
125	23	23	175	43	RCL
126	53	(	176	13	13
127	53	(	177	54	)
128	53	(	178	54	)
129	43	RCL	179	42	STO
130	01	01	180	26	26
131	45	YX	181	53	(
132	43	RCL	182	00	0
133	21	21	183	54	)
134	54	)	184	42	STO
135	65	×	185	09	09
136	43	RCL	186	43	RCL
137	23	23	187	26	26
138	54	)	188	42	STO
139	94	+/-	189	10	10
140	85	+	190	53	(
141	01	1	191	43	RCL
142	54	)	192	10	10
143	42	STO	193	55	÷
144	24	24	194	02	2
145	53	(	195	54	)
146	43	RCL	196	42	STO
147	01	01	197	10	10
148	65	×	198	53	(
149	53	(	199	43	RCL

LOC	CODE	KEY
200	19	19
201	85	+
202	43	RCL
203	10	10
204	54	)
205	42	STO
206	13	13
207	43	RCL
208	09	09
209	42	STO
210	27	27
211	71	SBR
212	16	A'
213	42	STO
214	15	15
215	43	RCL
216	12	12
217	42	STO
218	27	27
219	71	SBR
220	16	A'
221	42	STO
222	18	18
223	53	(
224	43	RCL
225	15	15
226	65	×
227	43	RCL
228	18	18
229	54	)
230	42	STO
231	19	19
232	53	(
233	00	0
234	54	)
235	32	X:T
236	43	RCL
237	19	19
238	22	INV
239	77	GE
240	02	02
241	46	46
242	43	RCL
243	12	12
244	42	STO
245	09	09
246	53	(
247	01	1
248	52	EE
249	04	4
250	94	+/-
251	54	)
252	32	X:T
253	43	RCL
254	10	10
255	22	INV
256	77	GE
257	02	02
258	62	62
259	61	GTO
260	01	01
261	90	90
262	43	RCL
263	09	09
264	91	R/S
265	76	LBL
266	16	A'
267	53	(
268	53	(
269	53	(
270	53	(
271	53	(
272	53	(
273	43	RCL
274	27	27
275	45	Y^X
276	43	RCL
277	21	21
278	54	)
279	65	×
280	43	RCL
281	23	23
282	54	)
283	94	+/-
284	85	+
285	01	1
286	54	)
287	45	Y^X
288	43	RCL
289	22	22
290	54	)
291	65	×
292	43	RCL
293	25	25
294	54	)
295	94	+/-
296	85	+
297	43	RCL
298	27	27
299	54	)
300	92	RTN

LRN

ACCELERATED $\ddot{x}_2$ (HP-34C)

The following program listing solves for the accelerated random input vibration level $\ddot{x}_2$ to accomplish *the same goal as PL-16*. PL-16 is written in BASIC language. PL-15 is written for the HP-34C programmable Calculator. The constants C_5 , $\ddot{x}_{2_{max}}$, b_1 thru b_4 must be calculated separately as shown below. Then their values are entered into PL-15.

INPUT PARAMETERS:

a_i (inches)	θ
$\ddot{x}_1$ (g rms)	Y
C_4 (ksi/g rms)	$\overline{A}$ (ksi)
T_1/T_2	$\overline{C}$ (ksi)
η	ΔK_c (ksi $\sqrt{in}$)

CONSTANTS:

$$b_1 = \frac{1}{a_i}\left[\frac{\Delta K_c}{C_5 Y}\right]^2$$

$$C_5 = \left(\frac{2\overline{A}}{\overline{C}}\right) C_4$$

$$b_2 = \left(\frac{1}{b_1}\right)^{\frac{\theta - 2}{2}}$$

$$b_3 = 1 - b_2\,\ddot{x}_1^{\,\eta(\theta - 2)}$$

$$b_4 = \ddot{x}_1\left(\frac{T_1}{T_2}\right)^{1/\eta\theta}\left(\frac{1}{b_3}\right)^{1/\eta\theta}$$

$$\ddot{x}_{2_{max}} = b_1^{\,1/2\eta}$$

HP-34C

$$f(x) = x - b_4 \left[1 - b_2 \, x^{\eta(\theta - 2)} \right]^{1/\eta\theta}$$

EXAMPLE:		HP-34C
	$b_2 = 1.1102214 \times 10^{-3}$ *	R_0
	$b_3 = 9.9888978 \times 10^{-1}$	R_1
	$b_4 = 7.952523$	R_2
	$\eta(\theta - 2) = 1.666$	R_3
	$1/\eta\theta = 3.0012005 \times 10^{-1}$	R_4
	$\ddot{x}_{2_{max}} \simeq 59.35586$	
$\ddot{x}_2 \equiv x$	x	R_5

$$f(x) = R_5 - R_2 \left[1 - R_0 \times R_5^{R_3} \right]^{R_4}$$

Two initial guesses: $\ddot{x}_1$; $\ddot{x}_{2_{max}}$ (i.e. 1; 59.35)

Program uses 17 lines. Refer to the following page for execution instructions. The solution $\ddot{x}_2$ for the above example is:

$$\ddot{x}_2 = 7.8691$$

* a_i = .007 inches for all of the above parameter values.

a_i > .004 inches for these equations to apply.

Turn On

Enter values of desired f(x) coefficients into $R_0 \rightarrow R_4$

Enter two initial guesses for x:

$\ddot{x}_1$

ENTER ↑

$\ddot{x}_{2_{max}}$ *

f SOLVE A

*The value for $\ddot{x}_{2_{max}}$ entered should be slightly less than the actual value of $\ddot{x}_{2_{max}}$. *Otherwise the program will take the ℓn of a negative* number (which is illegal) and "ERROR 0" will appear. EXAMPLE: $\ddot{x}_{2_{max}}$ = 3.0199. For $\ddot{x}_{2_{max}}$ enter the value of 3.01.

J. REFERENCES

1. Abramowitz, M., and Stegun, I., Handbook of Mathematical Functions, National Bureau of Standards, 10th Printing, December, 1972.

2. Papoulis, A., Probability, Random Variables, and Stochastic Processes, McGraw-Hill Book Co., New York, 1965.

I. SYMBOLS

$\Gamma(\alpha)$	Gamma Function with argument α
α	variable
x	dummy variable, variable
S	variable
Y	variable
$\gamma(\alpha,\tau)$	Incomplete Gamma Function with argument parameters α and τ.
$Q(\alpha,\tau)$	Incomplete Gamma Function with argument parameters α and τ.
y	variable
τ	variable
χ^2	variable
ν, ν_o	variable
$Q(\chi^2 \mid \nu)$, $P(\chi^2 \mid \nu)$	Incomplete Gamma Function Parameters
$Q_o(\alpha,\tau)$	Normalized Incomplete Gamma Function with argument parameters α and τ.
ϕ	interpolation variable
w	interpolation variable
$erf_p(\alpha)$, $erf(\alpha)$	error function defined by Papoulis [2]
$erf^{-1}(\alpha)$	inverse error function
$F(N)$	probability of failure in N stress cycles
z	variable
$g_1, \ldots g_{10}$	constants
K	variable