Natural Frequencies and Mode Shapes of Multi-Degrees of Freedom Systems on a Programmable Calculator

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Abstract—The Holzer tabulation method for determining the natural frequencies of multi-degree of freedom torsional systems is relatively easy to automate on a computer or a programmable calculator. The Holzer method has been extended to translational systems consisting of masses and springs configured so that the model starts with a mass and ends with a mass. For example, the method has been used to determine the natural frequencies of freight trains with an engine in the front and a caboose in the rear. The method presented here extends the basic Holzer theory further to accommodate lumped parameter structural models. A program is developed for a programmable calculator for determining the natural frequencies and mode shapes of multi-degree of freedom systems.

1. Holzer Tabulation Method

The Holzer tabulation method was developed for determining the natural frequencies of torsional multi-degree of freedom systems. Often, mechanical systems are equated to a shaft containing several disks, as shown in Fig. 1. The elasticity of the sytem is represented by an equivalent shaft that has the ability to store potential energy. The disks represent the equivalent mass moment of inertias of the system. If disk 1 in Fig. 1 is displaced through some angle θ while disk 4 is held stationary, energy is stored in the system. When the disks are released, the system will be set into torsional vibration at its set of natural frequencies. If there

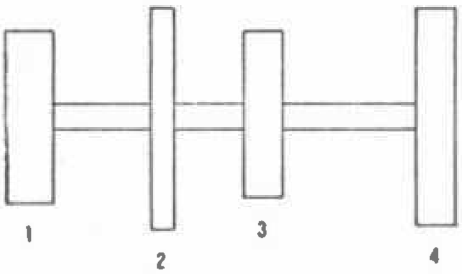


Fig. 1—Holzer Model of Torsional System.

is no damping, the system will continue to oscillate indefinitely without a forcing function.*

The Holzer tabulation method, shown in Table 1, is convenient for determining the natural frequencies (ω_n) and mode shapes. The natural frequencies are determined by assuming $\beta_1 = 1$ radian and trying various values of ω in the table. When the summation of torque is found to be zero for the system (Column 6, row N), a natural frequency is found. If the summation of torque is not zero, it is called residual torque. The residual torque can be plotted against various angular frequencies (ω) as shown in Fig. 2.

The Holzer table is used as follows:

1. Estimate or assume a value for ω
2. Calculate ω^2 from 1, above
3. Fill in Column 2 (I)
4. Fill in Column 7 (k_t)
5. For item 1 (first row)

Table 1—Holzer Tabulation Method (Assume Sample Values of ω and $\beta_1 = 1$ Radian)

1	2	3	4	5	6	7	8
Item	I	$I\omega^2$	β	$I\omega^2\beta$	$\sum_1^i I\omega^2\beta$	k_t	$\frac{1}{k_t}\sum_1^i I\omega^2\beta$
1	I_1		1			k_{t_1}	
2	I_2					k_{t_2}	
3	I_3					k_{t_3}	
:	:					:	
N	I_N						

Column 1 = Disk number
Column 2 = Mass moment of intertia, lb-in-sec^2
Column 3 = ω^2 multiplied by Column 2
Column 4 = β_i, relative angular displacement between disk i and disk 1, radians
Column 5 = Torque resulting from disk i, lb-in
Column 6 = Summation of torque, lb-in
Column 7 = Torsional spring constant k_t, in-lb/rad
Column 8 = The relative angle of twist between disks, radians θ

* For a derivation of the Holzer Method see C. R. Freberg and E. N. Kemler, *Elements of Mechanical Vibrations*, John Wiley & Sons, 1966, pp 72–8.

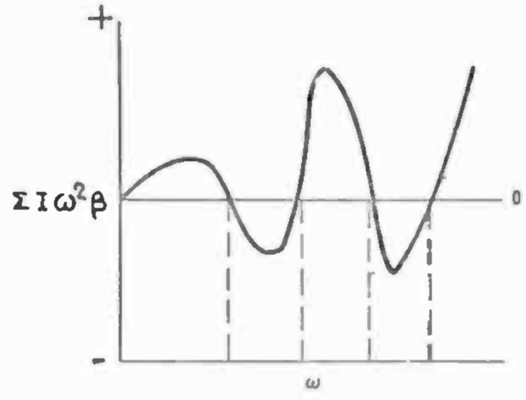


Fig. 2—Residual torque versus ω.

(a) Column 3, $I_1 \times \omega^2$
(b) Column 4, assume $\beta_1 = 1$ radian
(c) Column 5, torque (T) same as Column 3
(d) Column 6, (T) same as Column 5
(e) Column 8, (θ) Column 6_1 divided by Column 7_1

6. For item 2 (second row)
(a) Column 3, $I_2 \times \omega^2$
(b) Column 4, ($\beta_2 = \beta_1 - \Sigma T/k_t$) Column 4_1 − Column 8_1
(c) Column 5, (torque) Column $3_2 \times$ Column 4_2
(d) Column 6, (ΣT) Column 6_1 + Column 5_2
(e) Column 8, θ Column 6_2 divided by Column 7_2

7. For item N
(a) Column 3, $I_N \times \omega^2$
(b) Column 4, Column 4_{N-1} − Column 8_{N-1}
(c) Column 5, Column $3_N \times$ Column 4_N
(d) Column 6, (Residual Torque) Column 6_{N-1} + Column 5_N]

The system frequencies are found at the zero crossings of the residual torque plot. The residual torque curve can be very steep at the zero crossing points and care must be taken to accurately determine these points.

The Holzer tabulation method can be used for translational systems by substituting mass (Wt/g) for the mass moment of inertia I, translational spring constant K in lb/in for the torsional springs constants k_t, and the relative displacement of each mass from the first mass x in inches for β_i (x_1 is assumed to be 1 inch), as shown in Table 2.

A Holzer Structural model requires the last spring to be fixed to a foundation such as the earth, as shown in Fig. 3. To set this model into oscillation, an infinite force would be required. However, since this is only a mathematical model, we will set the entire system into oscillation and then remove the forcing function, so that the entire system vibrates at its set of natural frequencies and the summation of force equals zero.

Table 2—Translational System (W = weight, g = 386 in/sec^2, ω is in Hz, and x_1 assumed to be 1 inch)

1	2	3	4	5	6	7	8
Item	W/g	$W\omega^2/g$	x	$W\omega^2x/g$	$\Sigma W\omega^2x/g$	K	$1/K\, \Sigma W\omega^2x/g$
1	M_1		1			K_1	
2	M_2					K_2	
3	M_3					K_3	
4	M_4					K_4	
:	:					:	
N	M_N						

We will find that if we have not selected the proper value for ω the residual force will be infinite because the force generated by the last mass as shown in Column 5_N is equal to the infinite mass multiplied by ω^2x_N. Since we are looking for a zero crossing in the residual force versus angular frequency curve, we need only determine the sign of the relative displacement x_N. When x_N is positive, the residual force will be infinite (positive), and when x_N is negative the residual force will be infinite (negative). Therefore, a change in the sign of x_N is the result of a zero crossing and is found at a natural frequency of the system. Since Column 8 is the displacement between adjacent masses, mode shapes can be developed by determining the displacements between masses at the vibration modes.

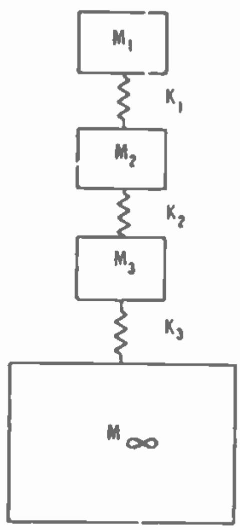


Fig. 3—Holzer structural model.

2. Use of Programmable Calculator

It is obvious that a large number of simple calculations are necessary to determine the natural frequencies and mode shapes of a multi-degree-of-freedom structural model. Since the calculations are repetitive, it is a simple job to program this problem for a computer or programmable calculator.

A program for a TI-58/59 programmable calculator has been developed. The program assumes ω to be 10 radians and runs through the Holzer tabulation calculations looking for a change in the sign of x_N. If x_N changes sign (plus to minus) between 0 and 10 radians, the program subtracts 5 radians from ω for averaging, divides by 2π, rounds the value to the nearest whole number and displays the answer as 1 Hz. If x_N does not change sign in 10 radians, the program will add 10 radians to ω and will repreat the above process. The angular frequency ω will be incremented by 10 radians until x_N changes sign. The calculator will then compute the frequency and display the results in Hz. The displacement between masses resides in the calculator memory and can be extracted for developing mode shapes.

The TI-58 contains enough memory to calculate the natural frequencies and mode shapes of a system containing up to seven masses and seven springs. The following description of the structural Holzer program is presented here to enable the reader to use it without mastering the art of programming calculators or computers. Before the details of the program are delineated, you will have to know a few things about the programmable calculator. The keyboard is shown in Fig. 4.

Besides the normal calculator functions, the following programming functions are required for this program:

LRN—(Learn)—Depressing this key allows the calculator to be programmed. Activating the key a second time will take the calculator out of the learn mode.

LBL A—(Label A)—Defines the start of this program.

STO—(Store)—Stores data in specific memory locations. For example, 10 STO 03 will store the number 10 in memory location 03.

RCL—(Recall)—Recalls the data from memory. For example, RCL 03 would bring the number 10 stored in location 03 to the display register.

SUM—(Sum)—Adds to a memory location. For example, 5 SUM 03 would add 5 to the contents of the memory at location 03.

Nop—(No Operation)—Provides spacing between program parts for later additions. Program execution simply performs no operation when this instruction is encountered. For example, the Nop function can be used to change the sign of a function by inserting a minus sign in place of the Nop. The address of the program step

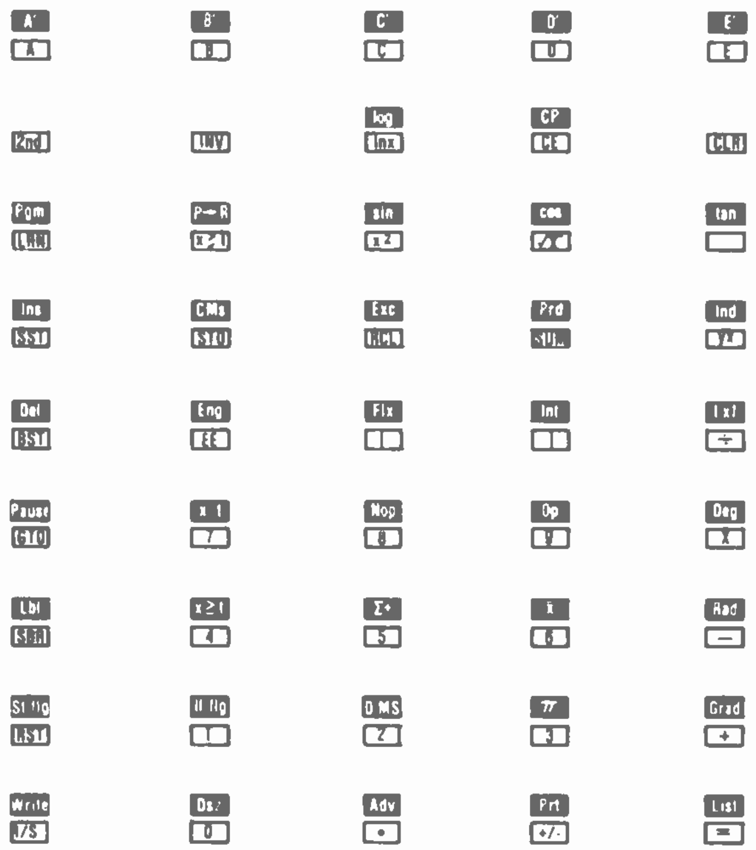


Fig. 4—Keyboard diagram for TI-58. There are two functions for most keys. The basic functions are shown as white keys with black lettering. The second functions, black rectangles with white lettering, are obtained by depressing the (2nd) key and then the key beneath the desired function.

must be remembered so that you can instruct the calculator to go to step XYZ (the address of Nop) and then press the (+/−) key.

GTO—(Go To)—This function is used to instruct the calculator to go to a specific address. It does this by moving its program pointer to the desired address. The program pointer is an internal device used by the calculator to determine which instruction it should perform next when executing a program. In the learn mode, the pointer automatically points to the next unfilled location in the program memory. When in the learn mode, depressing the (GTO) key and (A) key will tell the program pointer to go to the start of the program. If the calculator is not in the learn mode, the following key strokes, GTO, 1, 2, 5, LRN, will bring step 125 to the display register and place the calculator in the learn mode. The program can then be edited, e.g. the function of step 125 can be changed by depressing a different key.

FIX—Fixes the decimal point. For example, the following key strokes; 2nd, π, 2nd, FIX, 0 would result in the following display in this sequence; 0, 3.141592654, 3. Depressing INV, 2nd, FIX will restore 3.141592654 to the display.

INV—(Invert)—Inverts the function. For example, it was used to remove the fixed decimal in the above example.

$x \geq t$—(Test Instruction)—This is used as a conditional transfer. The test register (t) is set to zero in this program. This instruction is used to determine the change of sign of the last spring x_N. If $x_N \geq 0$ a transfer is made to the address specified. In this program, it goes to a program step which adds 10 radians to ω because the sign of x_N did not change. If x_N is less than the test register (0), it is negative and the next step in the program is followed, which is to recall the value of ω and display it in Hz.

R/S—(Run/Stop)—This function will start or stop the program. This instruction will halt the program and display the results of the last instruction.

RST—(Reset)—Resets the program pointer to step 0.

When the calculator is being programmed from the keyboard, the program step numbers are displayed each time a key is depressed. The number displayed is the program step of the next instruction to be entered.

3. Three-Mass, Three-Spring Structural Holzer Program

The details of a three-mass three-spring structural Holzer program for a TI-58/59 are described below. The following memory locations are preassigned to the variable ω, the masses, and springs of Fig. 3:

Memory Location	Contents
01	ω_1, radians/sec
02	M_1, first weight, lbs
03	M_2, second weight, lbs
04	M_3, third weight, lbs
05	K_1, first spring constant, lb/in
06	K_2, second spring constant, lb/in
07	K_3, third spring constant, lb/in

The program for the calculator is as follows.

Key Strokes	Function
LRN	Enter learn mode
2nd Lbl A	Defines start of program
RCL 1	Recalls ω
x^2	Calculate ω^2
STO 8	Stores ω^2 in memory location 08
(RCL 8 × RCL 2 ÷ 386) STO 9	Calculate Force (F) on first spring

	($\omega^2 x_1 W_1/g$) where x_1 is assumed to be 1 inch and stores in memory location 09
(1 − (RCL 9 ÷ RCL 5)) STO 10	Calculate x_2 by determining the displacement between M_1 and M_2, which is the force on M_1 divided by K_1 (RCL 9 ÷ RCL 5) and subtracting it from x_1 (one inch). Stores at memory location 10.
(RCL 8 × RCL 3 × RCL 10 ÷ 386) SUM 9	Calculate Σ F by computing the force on the second spring ($\omega^2 W_2 x_2/g$) caused by M_2 and summing it to the force on the first spring.
(RCL 10 − (RCL 9 ÷ RCL 6)) STO 11	Calculate x_3 by computing the displacement between M_2 and M_3, which is the total force on the second spring divided by K_2, subtracted from x_2. Stored at memory location 11.
(RCL 8 × RCL 4 × RCL 11 ÷ 386) SUM 9	Calculate Σ F by computing the force on the third spring ($\omega^2 W_3 x_3/g$) caused by M_3 and summing to memory location 9
(RCL 11 − (RCL 9 ÷ RCL 7))	Calculate x_N by computing the displacement between M_3 and $M\infty$ which is the total force on spring 3 divided by K_3, subtracted from x_3. A change in sign of this displacement indicates a zero crossing.
2nd Nop 2nd x ≥ t 175 (see note at end of program)	Conditional transfer if $x_3 \geq 0$, go to 175 (move the program pointer to step 175). If $x_3 \leq 0$ proceed to the next step.
((RCL 1 − 5) ÷ (2 × 2ndπ))	Calculate frequency in Hz. Recall ω from memory location 01, subtract 5 radians (for averaging within the 10 radian steps) and divide by 2π.
2nd FIX O R/S	Fixes decimal place to nearest whole number, stops program and displays frequency in Hz.
LRN	Exit learn mode
GTO 175 (see note at end of program)	Moves program pointer to Location 175 (an arbitrary urused memory address)

LRN	Enter learn mode at program step 175
10SUM 1	Adds 10 radians to ω
GTO A	Moves program pointer to the start or program (step 0) to recalculate x_N with a new ω (old ω + 10 radians)
LRN	Exit learn mode, End of Program

Note: The conditional transfer (2nd Nop $x \geq$ t 175) contains a Nop instruction and a transfer address of 175. The transfer address was arbitrarily selected as program step number 175 since the program for the three-mass three-spring system is only 112 steps long not including the 6 additional program steps (1, 0, SUM, 1, GTO, A) beginning at the transfer address 175. When programming the calculator for a six or seven spring/mass system, step 175 would already have been used before getting to the conditional transfer instruction. The transfer address must be greater than the program step number of the Nop instruction. The transfer address can be any step number greater than 212 and less than 234 for the TI-58 for any system containing up to 7 springs and 7 masses. The Nop instruction is inserted in the program prior to the test instruction x ≥ t. This test instruction was used to determine the change in sign (positive to negative) of the relative displacement of the last spring, x_N, which determined the frequency of the first mode of vibration. The second mode will be found when x_N changes sign again.

In determining the frequency of the second mode the blank instruction (Nop) is replaced with (INV) so that the conditional transfer instruction is INV $x \geq t$ 175. The program will now look for x_N to change from negative to positive in determining the frequency of the second mode. The next mode is found by replacing INV with Nop.

The operation of the calculator is as follows. First place the following values in memory:

10 STO 01	start with 10 rads.
M_1 STO 02	Weight No. 1
M_2 STO 03	Weight No. 2
M_3 STO 04	Weight No. 3
k_1 STO 05	Spring No. 1
k_2 STO 06	Spring No. 2
k_3 STO 07	Spring No. 3

Depress A

Calculator will compute the 1st mode in Hz

GTO	2nd MODE
91	(LOCATION of Nop)
LRN	

INV
LRN
RST
A (SECOND MODE COMPUTED)
GTO 3rd MODE
91
LRN
2nd
Nop
LRN
RST
A (THIRD MODE COMPUTED)

The mode shapes can be developed by determining the displacements between the masses. The displacement between masses is found in Column 8 of the Holzer table $K^{-1} \Sigma W\omega^2 x/g$. Note that $\Sigma W\omega^2 x/g$ is always stored in memory location 9 for each mass and the displacements can be found by dividing it by the springs constant K_i. Looking at the program we find that RCL 10 = (1 − (RCL 9 ÷ RCL 5)) for mass 1 where memory location 9 contained $\Sigma_1^1 W^2 x$ at the time of the calculation and memory location 5 contains K_1. Since the displacement between M_1 and $M_2(\delta_{1-2})$ is equal to $K^{-1} \Sigma_1^1 W\omega^2 x$, we obtain RCL 10 = $(1 - \delta_{1-2})$ and $\delta_{1-2} = 1 -$ RCL 10.

The displacements might be very small; therefore, the calculator must be taken out of its fixed decimal place mode. After the calculator displays the frequency of the first mode of vibration, the following keys are depressed:

INV, 2nd, FIX
1 − RCL 10 =

The relative displacement between mass 1 and mass 2 will be displayed on the calculator.

The displacement between masses 2 and 3 can be determined in a similar manner.

RCL 11 = (RCL 10 − (RCL 9 ÷ RCL 6))

δ_{2-3} = RCL 9 ÷ RCL 6

where RCL 9 = $\sum_1^2 W\omega x/g$

RCL 6 = K_2

δ_{2-3} = RCL 10 − RCL 11

The displacement between masses 3 and ∞ is determined by:

$\delta_{3-\infty}$ = RCL 9 ÷ RCL 7

$$\text{where RCL 9} = \sum_{1}^{3} W\omega^2 x/g$$

$$\text{RCL 7} = K_3$$

The displacements between masses at the higher modes are obtained by following the procedure outlined above remembering always to remove the calculator from the fixed decimal mode.

The displacements between masses in each mode are normalized so that the maximum displacement between adjacent masses (δ_{1-2}, δ_{2-3}, $\delta_{3-\infty}$) is a unit deflection. This is accomplished by dividing all displacements by the largest displacement. The displacements may be plotted to delineate the mode shape.

4. Example

Fig. 5 shows a steel rack with three shelves supporting rigid masses. The rack structure is welded so that the shelves and top are fixed to the four columns. The base of the assembly is firmly fixed to a vibration table. The problem is to find the natural frequencies and mode shapes that would be found if the table were to oscillate in the x direction as shown on the figure.

The Holzer lumped parameter structural model was shown earlier in Fig. 3. The spring constants K represent the compliance of the structure between the masses. Values of the masses are shown in Fig. 5. The rack weight apportioned to each mass is 0.24 the column weight plus the shelf weight, or approximately 10 lbs/mass. Therefore, M_1 = 110, M_2 = 160, and M_3 = 210 lbs.

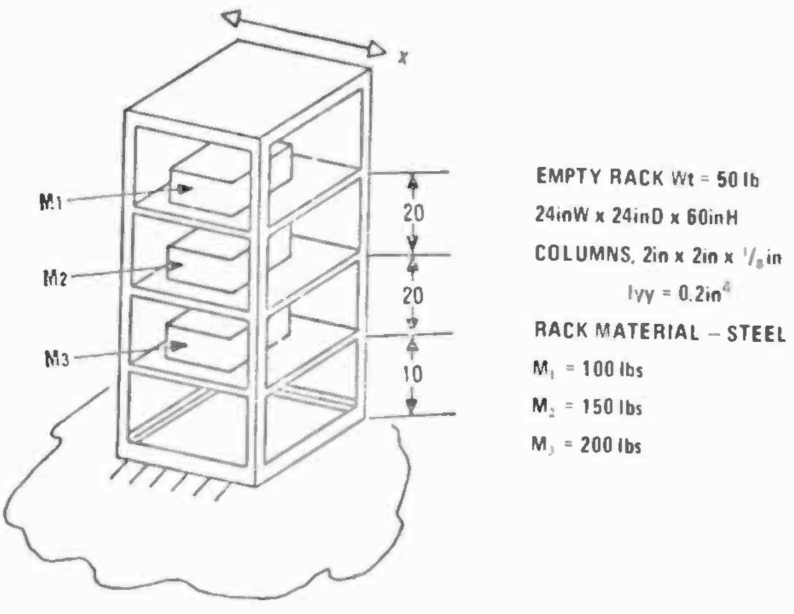


Fig. 5—Rack assembly used for example.

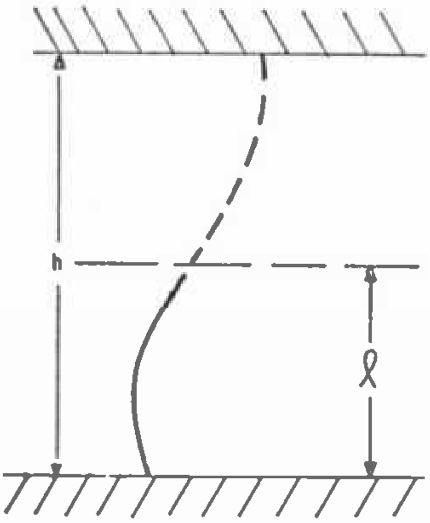


Fig. 6—Deflection curve.

The spring constants K are derived from a static analysis of the structure. If the rack was deflected in the x direction, the columns between the shelves would bend in an "s" shape with the ends perpendicular to the shelves. The deflection curve of the columns looks like two cantilever beams in series, as shown in Fig. 6, where h is the distance between shelves ($h = 2l$). The spring constant of a cantilever beam with length l is

$$K = \frac{3EI}{l^3}$$
$$= \frac{3EI}{(h/2)^3} \text{ per beam.}$$

There are 8 beams per shelf, and springs in series add like capacitors in series:

$$K_{column} = \frac{1}{\dfrac{1}{K_{beam_1}} + \dfrac{1}{K_{beam_2}}} = \frac{1}{\dfrac{1}{24EI/h^3} + \dfrac{1}{24EI/h^3}} = 12\,EI/h^3$$

$$K_{4c} = 48\,EI/h^3 \text{ (per shelf)}$$

$$K = \frac{48EI}{h^3}$$

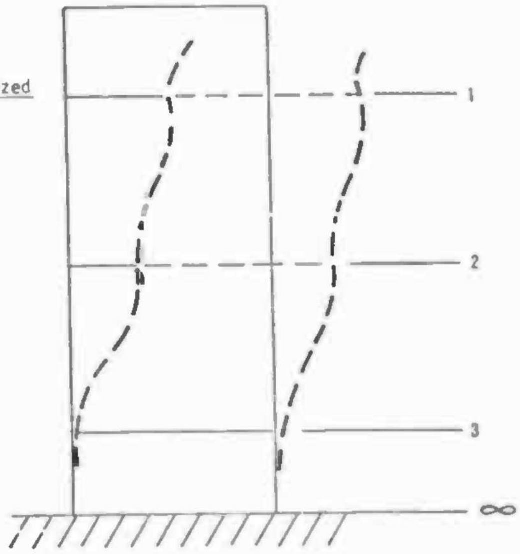
$$K_1 = K_2 = \frac{48(30 \times 10^6)(0.2)}{(20)^3} = 36{,}000 \text{ lb/in}$$

$$K_3 = \frac{48(30 \times 10^6)(0.2)}{(10)^3}$$

$$K_3 = 288{,}000 \text{ lb/in} = 288{,}000 \text{ lb/in.}$$

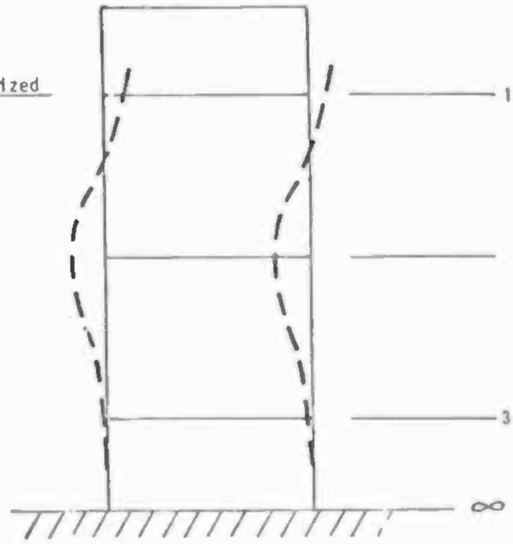
MODE 1, 31 Hz

	Key Stroke	Display	Normalized
δ_{1-2} =	(1-RCL10)	= 0.317	0.502
δ_{2-3} =	(RCL10-RCL11)	= 0.631	1.000
$\delta_{3-\infty}$ =	(RCL9 ÷ RCL7)	= 0.083	0.131



MODE 2, 79 Hz

	Key Stroke	Display	Normalized
δ_{1-2} =	(1-RCL10)	= 1.979	1.00
δ_{2-3} =	(RCL10-RCL11)	= 0.839	-.427
$\delta_{3-\infty}$ =	(RCL9 ÷ RCL7)	= -.171	-.086



MODE 3, 125 Hz

	Key Stroke	Display	Normalized
δ_{1-2} =	(1-RCL10)	= 4.94	0.211
δ_{2-3} =	(RCL10-RCL11)	= -23.37	-1.00
$\delta_{3-\infty}$ =	(RCL9 ÷ RCL7)	= 19.99	0.855

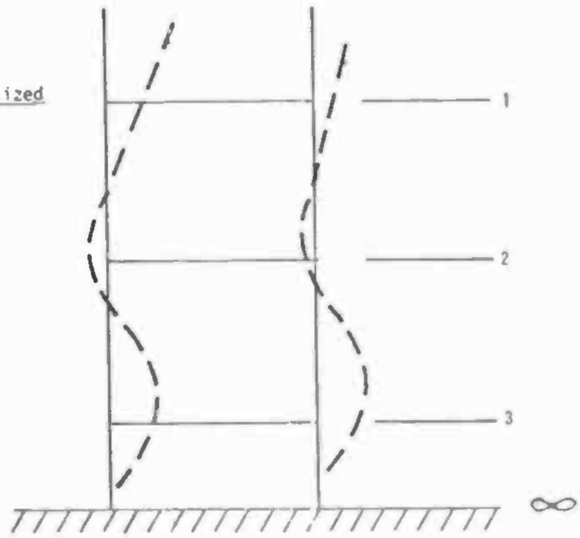


Fig. 7—Deflection curve plotted (a) for mode 1, 31 Hz, (b) for mode 2, 79 Hz, and (c) for mode 3, 125 Hz.

Once the calculator is programmed, the data is loaded as follows:

	Key Strokes	
ω	10	STO 01
M1	110	STO 02
M2	160	STO 03
M3	210	STO 04
K1	36,000	STO 05
K2	36,000	STO 06
K3	288,000	STO 07

The natural frequencies are now computed with the following sequence:

Key Stroke	Display	
A	31Hz	
		1st mode
GTO		
91		
LRN		
INV		
LRN		
RST		
A	79 Hz	2nd mode
GTO		
91		
LRN		
2nd		
Nop		
LRN		
RST		
A	125 Hz	3rd mode

Before the mode shapes are determined, the calculator must be taken out of the whole integer mode with the following key strokes: INV, 2nd, and FIX. The mode shapes are determined by computing the deflections between masses, normalizing to unit deflection, and plotting, as shown in Fig. 7(a–c).