PORTABLE COMPUTER WORLD

Time marches on and Calculator Corner must march with it. From now on this column will be called PORTABLE COMPUTER WORLD, so that not only calculators but new machines like the Sharp PC1500 and the HP-75C and Epson HX20 can be covered without hurting anyone's feelings. Just to prove that I don't intend to forget calculators, this months column, by Ed Rosenstiel, is for the TI-59

CONTINUED FRACTIONS

In my time continued fractions were not done at school, so I'll explain briefly this remarkably easy to understand concept which may well have been known in antiquity, and which has ramifications in many branches of higher mathematics.

Any (real) number X has a unique counterpart of the form

$$N_1 + \cfrac{1}{N_2 + \cfrac{1}{N_3 + \cfrac{1}{\text{etc}}}}$$

with integral Ns, either finitely many — when trivially $X = N_1$ or when X is just a common fraction, ie, rational — or else the 'continued fraction' (technically called 'simple' since all its 'numerators' are 1) goes on for ever, like an infinite series.

Take, for example, $X = 2.\dot{2}8571\dot{4}\ldots$ which any schoolboy will tell you is equal to 16/7. To work out its CF:

$$16/7 = 2 + 2/7 = 2 + \frac{1}{7/2} = 2 + \frac{1}{3+½} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + 1/1}}$$

which shows that periodic decimals (being rational) have *finite* CFs. If, however, you start with a square root like $\sqrt{2}$ we have:

$$\sqrt{2} \simeq 1.414236\ldots = 1 + \cfrac{1}{\cfrac{10000000}{4142136}} = 1 + \cfrac{1}{2 + \cfrac{1715728}{4142136}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{4142136}{1715728}}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{710678}{1715728}}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}$$

Considering that we started with an (8-digit) *approximation* it is a fair guess (and true) that the twos go on *ad infinitum.*— ie, we have a *periodic* CF with a 1-digit period, usually abbreviated as

$\sqrt{2} = [1;\overline{2}\ ]$.

In fact, all square roots of *integers* are periodic CFs, eg:

$\sqrt{3} = [1;\overline{1,2}\ ]$

$\sqrt{13} = [3;\overline{1,1,1,1,6}]$

An important use of CFs follows from the property that, whenever one truncates an infinite CF after any number of terms, a common fraction results which is a 'best' approximation to the infinite CF. Eg,

$$\sqrt{2} \simeq 1 + 1/2 = 1.5$$

$$\simeq 1 + \frac{1}{2 + 1/2} = 1.4$$

$$\simeq 1 + \cfrac{1}{2 + \cfrac{1}{2 + 1/2}} = 1 + \frac{1}{2 + 7/5}$$

$$= 1 + \frac{10}{24} \simeq 1.417, \text{ etc.}$$

The first of two TI-59 programs is based on a formula of Patz (1941). It displays the list representing the CF of $\sqrt{N}$ (called the list of 'partial quotients') whenever the N is entered and followed by keystroke A. All these lists will be periodic after a certain point, and some periodic CFs will also show other striking regularities first proved by the French mathematician Lagrange (1766).

The simplest periodic CF is $[1;\overline{1}\ ] = X$, say.

Then

$$X = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \text{etc}}} = 1 + \cfrac{1}{1 + \cfrac{1}{x}} = 1 + \frac{x}{x+1}$$

Continued Fraction of $\sqrt{X}$

000	76	LBL	032	43	RCL
001	11	A	033	04	04
002	99	PRT	034	95	=
003	98	ADV	035	59	INT
004	42	STO	036	99	PRT
005	01	01	037	66	PAU
006	34	√X	038	42	STO
007	42	STO	039	05	05
008	02	02	040	65	×
009	59	INT	041	43	RCL
010	42	STO	042	04	04
011	03	03	043	75	-
012	99	PRT	044	43	RCL
013	66	PAU	045	03	03
014	43	RCL	046	95	=
015	01	01	047	42	STO
016	75	-	048	03	03
017	43	RCL	049	33	X²
018	03	03	050	94	+/-
019	33	X²	051	85	+
020	95	=	052	43	RCL
021	42	STO	053	01	01
022	04	04	054	95	=
023	76	LBL	055	55	÷
024	12	B	056	43	RCL
025	43	RCL	057	04	04
026	02	02	058	95	=
027	85	+	059	42	STO
028	43	RCL	060	04	04
029	03	03	061	61	GTO
030	95	=	062	12	B
031	55	÷	063	00	0

Enter X and press A

Inverse CF Program to recover X from the Continued Fraction of $\sqrt{X}$

000	76	LBL	032	42	STO	061	13	C	090	12	B	119	91	R/S
001	15	E	033	13	13	062	42	STO	091	67	EQ	120	61	GTO
002	99	PRT	034	65	×	063	16	16	092	14	D	121	11	A
003	98	ADV	035	43	RCL	064	65	×	093	42	STO	122	76	LBL
004	42	STO	036	12	12	065	43	RCL	094	17	17	123	13	C
005	10	10	037	85	+	066	14	14	095	65	×	124	43	RCL
006	91	R/S	038	43	RCL	067	85	+	096	43	RCL	125	12	12
007	76	LBL	039	10	10	068	43	RCL	097	12	12	126	55	÷
008	16	A'	040	95	=	069	12	12	098	85	+	127	43	RCL
009	67	EQ	041	66	PAU	070	95	=	099	43	RCL	128	11	11
010	13	C	042	99	PRT	071	66	PAU	100	14	14	129	95	=
011	42	STO	043	42	STO	072	99	PRT	101	95	=	130	47	CMS
012	11	11	044	14	14	073	42	STO	102	66	PAU	131	99	PRT
013	65	×	045	43	RCL	074	12	12	103	99	PRT	132	98	ADV
014	43	RCL	046	13	13	075	43	RCL	104	42	STO	133	91	R/S
015	10	10	047	65	×	076	16	16	105	14	14	134	76	LBL
016	85	+	048	43	RCL	077	65	×	106	43	RCL	135	14	D
017	01	1	049	11	11	078	43	RCL	107	17	17	136	43	RCL
018	95	=	050	85	+	079	15	15	108	65	×	137	14	14
019	42	STO	051	01	1	080	85	+	109	43	RCL	138	55	÷
020	12	12	052	95	=	081	43	RCL	110	11	11	139	43	RCL
021	66	PAU	053	42	STO	082	11	11	111	85	+	140	15	15
022	99	PRT	054	15	15	083	95	=	112	43	RCL	141	95	=
023	43	RCL	055	99	PRT	084	42	STO	113	15	15	142	47	CMS
024	11	11	056	98	ADV	085	11	11	114	95	=	143	99	PRT
025	99	PRT	057	91	R/S	086	99	PRT	115	42	STO	144	98	ADV
026	98	ADV	058	76	LBL	087	98	ADV	116	15	15	145	91	R/S
027	91	R/S	059	11	A	088	91	R/S	117	99	PRT	146	00	0
028	76	LBL	060	67	EQ	089	76	LBL	118	98	ADV	147	00	0
029	17	B'												
030	67	EQ												
031	14	D												

Enter in sequence the numbers (= "partial quotients") of the CF of $\sqrt{X}$, pressing after each of the first five entries one of E, A',B',A or B (in that order).Next alternate A & B. For approximations of X enter 0 and repeat the last keystroke.

hence (X – 1) (X + 1) = X, ie, x^2-x-1=0
Of this equation

$$X = \sqrt{5/2} + 1/2 \simeq 1.62$$

is the relevant root, since X>1. Many will recognise this equation as the one for the famous 'Golden Section', which defines a rectangle with sides 1 and ≃1.62 respectively, and that the so-called 'convergents' of [1;1], ie, 1/1, 2/1, 3/2, 5/3, 8/5, contain the well-known Fibonacci numbers. (This example also shows that there are periodic CFs which are not just the square root of an integer.)

Having loaded the CF-program into the TI-59, any *real* number can also be entered — eg, π^2 (with $\pi \simeq 3.141592654$), which gives the CF of $\sqrt{\pi^2}$ = CF of π = [3;7,15,1,292,1,1,1,2,1,· · · ·] accurately to 10 places. Although unending, no periods or regularities have been discovered among the first several thousand partial quotients of this CF, nor in any other irrational reals excepting square roots *and* the Euler number e = INV 1n 1 and some simple arithmetical formulas based on these two exceptions. (Enter e^2 and discover a 'regular' non-periodic infinite CF!)

The CF of another Euler number called γ (gamma) = the limit (as n — ∞) of (1 + 1/2 + 1/3 + . . . + 1/n – 1n n) ≃ 0.577. . . has also been calculated to several thousand digits without finding any regularities. This makes it likely by unproven that this number is not rational, but here is one of the famous unsolved problems of mathematics, namely whether γ (gamma) is the root of some algebraic equation or is transcendental like e and π , or is rational after all.

Now to use the inverse CF-program: Enter in sequence the partial quotients of some CF by pressing after each of the first five the keys E,A',B',A,B, respectively, then follow further entries by alternating between keys A and B. After any entry and appropriate key stroke the corresponding convergent is displayed, first its numerator, then the denominator. After the first two entries (E and A'), whenever zero is entered and followed by the keystroke which was used last, the decimal value of the convergent reached so far is displayed and the program is reset.

Examples

a) $\pi \simeq$ [3;7,. .].
Enter 3,E; display 3.
enter 7,A';display 22(PAU/PRT)7
enter 0,A';display 3.142857143

b) $\pi \simeq$[3;7,15,1,292,. . .]
Enter 3,E; display 3.
enter 7,A';display 22(PAU/PRT) 7
enter 15,B'; display 333(PAU/PRT)106
enter 1,A; display 355(PAU/PRT)113
enter 0,A; display 3.14159292

It is noteworthy that the approximation $\pi \simeq 355/113$, known already in China in antiquity, which is accurate to 2.7×10^{-7} is followed by the unusually large partial quotient 292. Such large PQs in CFs often give a clue to hidden and obscure interrelationships (Churchhouse, 1973).

Regarding the (so far unending) CF of γ (gamma) it is — as for the CF of π — not even known whether the partial quotients have an upper bound, but that they are unbounded for e was already known to Euler (1701-1783). Finally, the convergents of $\sqrt{N}$ readily supply integer solutions of the famous PELL equation $X^2 - N.Y^2 = 1$ (Beiler, 1964) — but that is another story.

References

Beiler, A H: *Recreations in the Theory of Numbers* Dover Publications, Inc., New York, 1964.
Churchhouse RF: *JIMA, 1973,9,17*
Patz W *Tafel der regelmassigen Kettenbruche,* Becker & Erler, Leipzig, 1941.